Independent Study
in Infinite Graph Theory

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1 Introduction

In this paper we will prove some results about infinite graphs. We show that for every linear order there is a graph with a distinguished vertex such that the edges adjacent to that vertex have the given order in any plane drawing. The other results are concerned with connectivity. We prove a generalization of a characterization of 2-connected graphs and prove that k-connectedness does not imply the existence of finite k-connected subgraphs for $k > 2$.

2 Embedding linear orders in edge sequences

Our goal in this section is to construct for every countable, linear order $(D, \leq)$ a planar graph G with a distinguished vertex v such that for every plane drawing of G the ordering of the edges incident to v is isomorphic to $(D, \leq)$. The *ordering* of edges incident to v is defined as follows: Sitting on v looking north and turning around once (in either direction, since we consider drawings up to homeomorphisms of the plane) an edge e_1 is smaller than edge e_2 if it appears before e_2 on the list of edges.

To control the different plane drawings of a graph we will consider 3-connected graphs. 3-connected graphs have exactly one plane drawing up to equivalence. For exact defintions see [1]. Thus, if we exhibit a plane drawing of a 3-connected graph with the desired property, then every plane drawing of that graph has the property.

To prove that our (infinite) graph G is indeed 3-connected we extend the following characterization of 3-connected finite graphs to infinite graphs.

Theorem 1. *A finite graph G is 3-connected if and only if there exists a sequence $G_0, \dots, G_n$ of graphs with the following properties:*

1. $G_0 = K^4$ *and* $G_n = G$,

2. G_{i+1} *has an edge* xy *with* $d(x) \geq 3$, $d(y) \geq 3$ *and* $G_i = G_{i+1}/xy$, *for every* $i < n$.

Proof. See [1]. $\square$

Another theorem, that will be utilized in the next proof and throughout the paper is a global version of Menger's Theorem.

Theorem 2. *A graph is k-connected if and only if it contains k independent paths between any two vertices.*

Proof. See [1]. $\square$

Now, we can formulate a corollary tailored of 1 for our main theorem.

Corollary 1. *Let $G_0, G_1, \ldots$ be an infinite sequence of graphs with the following properties:*

1. G_0 *as shown in Figure 1*

2. *there is a path* xyz *in* $G_i = (V_i, E_i)$ *and* $G_{i+1} = (V_i \mathbin{\dot\cup} \{v\}, E_i \mathbin{\dot\cup} \{xv, yv, zv\})$.

Then $G = \bigcup_{i=0}^{\infty} G_i$ *is 3-connected.*

Proof. First, we prove that every G_i is 3-connected and from that, we conclude that G is 3-connected.
We tackle the first task by showing inductively that for every G_i there is a sequence of graphs as in Theorem 1. For G_0 the sequence is $K^4 = G_{-2}, G_{-1}$ as shown in Figure 1. Now assume that $G_0, \ldots, G_i$ is a sequence satisfying the requirements of Corollary 1. We have a path xyz in $G_i = (V_i, E_i)$ and $G_{i+1} = (V_i \mathbin{\dot\cup} \{v\}, E_i \mathbin{\dot\cup} \{xv, yv, zv\})$ and therefore $d(y) \geq 3$, $d(v) \geq 3$ in G_{i+1}. Furthermore, $G_i \simeq G_{i+1}/yv$ and hence, there is also a sequence of graphs ending in G_{i+1} that satisfies the conditions of Theorem 1.

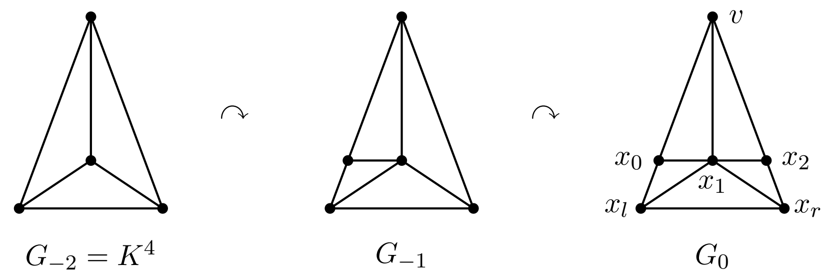

Figure 1: G_{-2}, G_{-1}, G_0

For the second task we assume that G is not 3-connected. Then, by Theorem 2, there are two vertices of G not having 3 independent paths between them. Pick a G_i that contains both vertices. Since it is 3-connected, there are 3 independent paths between them in G_i and thus in G. Contradiction. $\square$

Thus, we have a tool at hand to construct an infinite 3-connected graph.

Theorem 3. *Let $(D, \leq)$ be a countable, linear order. There exists a planar graph G and a vertex v of G such that in every plane drawing of G the ordering of the vertices incident to v is isomorphic to $(D, \leq)$ up to homeomorphisms of the plane.*

Proof. By $<$ we denote the strict version of $\leq$, that is $d < d'$ if and only if $d \leq d'$ and $d \neq d'$. Let $d_0, d_1, d_2, \ldots$ be a fixed enumeration of D. We inductively construct the Graph G as the limit of a sequence of graphs $G_2 \subset G_3 \subset G_4 \subset \ldots$ with vertex v and an isomorphism f between D and the edges incident to v.
Without loss of generality we assume that $d_0 < d_1 < d_2$. We start with G_2 and $f(d_0) = vx_0$, $f(d_1) = vx_1$ and $f(d_2) = vx_2$ as in Figure 1.
Now, consider a graph $G_i = (V_i, E_i)$, f with domain $D_i = \{d_0, \ldots, d_i\}$ and the next element d_{i+1}. There are three cases: If $d_{i+1} < d_k$ for all $k \in \{0, \ldots, i\}$, let $d_{min} = \min D_i$. Pick the path $vf(d_{min})x_l$ and set $G_{i+1} = (V_i \dot\cup \{w\}, E_i \dot\cup \{vw, f(d_{min})w, x_l w\})$ and $f(d_{i+1}) = w$ with a new vertex w. Analogously, if $d_{i+1} > d_k$ for all $k \in \{0, \ldots, i\}$ let $d_{max} = \max D_i$. Pick the path $vf(d_{max})x_r$ and set $G_{i+1} = (V_i \dot\cup \{w\}, E_i \dot\cup \{vw, f(d_{max})w, x_r w\})$ as in Corollary 1 and $f(d_{i+1}) = w$. Otherwise, there are $d_l, d_r \in D_i$ such that $d_l < d_{i+1} < d_r$ and there is no $d \in D_i$ such that $d_l < d < d_{i+1}$ or $d_{i+1} < d < d_r$, i. e. d_l and d_r are the predecessor and successor of d_{i+1} restricted to D_i. Pick the path $f(d_l)vf(d_r)$ and set $G_{i+1} = (V_i \dot\cup \{w\}, E_i \dot\cup$

$\{f(d_l)w, vw, f(d_r)w\}$) as in Corollary 1 and $f(d_{i+1}) = w$ with a new vertex w.

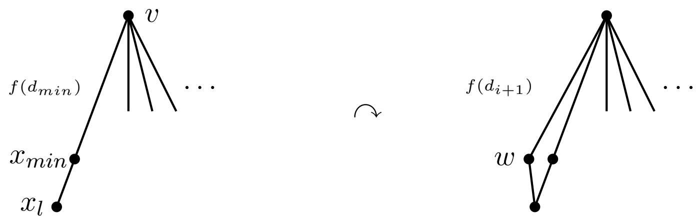

Figure 2: From G_i to G_{i+1}, case $d_{i+1} < d_k$ for all k

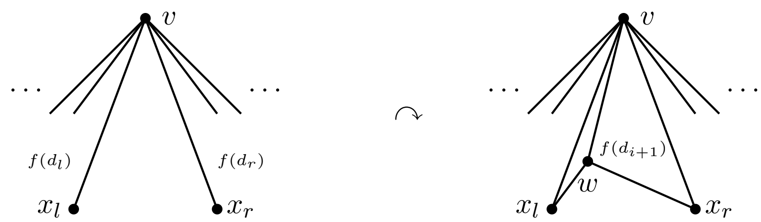

Figure 3: From G_i to G_{i+1}, case $d_l < d_{i+1} < d_r$

The graph $G = \bigcup_{i=0}^{\infty} G_i$ is 3-connected by Corollary 1. It is also planar: G_0 is obviously planar and every vertex addition according to Corollary 1 retains planarity, as long as the two edges of the path bound the same face of the given graph. This is true for all cases above. Thus, G is planar. It remains to show that the ordering of the edges incident to v in any plane drawing of G is ismorphic to $(D, \leq)$. Again, this is true for the *intended* drawing above. Since drawings of 3-connected planar graphs are unique up to homeomorphims of the plane, the ordering of the edges will also be equivalent up to these homeomorphisms. $\square$

3 Connectedness of Infinite Graphs

In this section we are concerned with connectedness of infinite graphs and their finite subgraphs. We begin with a characterization theorem for 2-connected graphs which generalizes the following theorem for finite graphs.

Theorem 4. *A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.*

Proof. See [1]. $\square$

To simplify the proof of our result, we begin with an easy Lemma.

Lemma 1. *Let G be an infinite, 2-connected graph, G' a finite, 2-connected subgraph of G and $v \in G - G'$. Then, there exists a G'-path P in G containing v. Furthermore, $G' \cup P$ is again finite and 2-connected.*

Proof. Let $v_1 \in V(G')$ and P_1, P_2 two independent v_1-v paths in G. The existence of these paths is guaranteed by Theorem 2. For $j = 1, 2$ let P_j' be the suffix of P_j such that $\mathring{P}_j$ does not contain any vertex from G'. We consider two cases.
If $P_1' = P_1$ and $P_2' = P_2$, then pick $v_2 \in V(G') \backslash \{v_1\}$ and a v_2-v path P_3. If P_3 meets G', then let P_3 be the suffix that meets G' only in its first vertex. Note that we can pick P_3 so that it does not begin in v_1, since there are two disjoint v_2-v paths and only one of them can contain v_1. If P_3 and $P_1 \cup P_2$ are disjoint, then choose $P := P_1 \cup P_3$. Otherwise, let v' be the first vertex of P_3 that is in $P_1 \cup P_2$. Without loss of generality assume $v' \in P_1$. Then choose $P := P_3 v' P_1 \cup P_2$.
For the second case we assume without loss of generality that $P_1 \neq P_1'$. Then, the first vertices of P_1' and P_2' are not equal and we choose $P := P_1' \cup P_2'$.
P is in either case a G'-path containing v and the 2-connectedness of $G' \cup P$ follows easily from Theorem 4. $\square$

Now, we can extend this well-known characterization of finite 2-connected graphs to infinite graphs.

Lemma 2. *Let G be an infinite graph. G is 2-connected if and only if it can be constructed from a cycle by countably often adding H-paths to graphs H already constructed.*

Proof. Let G be constructed as described above and assume that it is not 2-connected. By Theorem 2, there are vertices v, w in G that are not connected by two independent paths. Now consider the finite subgraph of G that is obtained by constructing G until v and w are added. This graph is 2-connected by Theorem 4. Hence, there are two independent paths in G that connect v and w. Contradiction.
For the other direction we will construct an infinite sequence $G_0, G_1, G_2, \dots$ of subgraphs of G such that G_0 is a cycle, $G_{i+1} = G_i \cup H_i$ for some G_i-path

P_i and $G = \bigcup_{i=0}^{\infty} G_i$. Let $v_0, v_1, v_2, \dots$ be a fixed enumeration of $V(G)$.
To begin, we choose G_0 to be a circle in G containing v_0 and v_1. Now, given G_i, let v_k be the smallest vertex (with respect to the fixed enumeration) that is not in G_i. If there is an edge $e = \{v_m, v_n\}$ such that $m, n < k$ and $e \in G - G_i$, then let $G_{i+1} := G_i + e$. Note that such an edge is a G_i-path. If there is no such edge, we apply Lemma 1 which gives us a G_i-path P_i containing v_k and let $G_{i+1} := G_i \cup P_i$.
It remains to show that $G = \bigcup_{i=0}^{\infty} G_i$. Obviously, the construction guarantees $G \supseteq \bigcup_{i=0}^{\infty} G_i$.
Let $v \in V(G)$. If $v \in G_0$ we are done. Otherwise, $v \in P_i$ for some i, either because it is in a P_i picked for some smaller node w or it was the smallest node at some stage, in which case P_i was picked to add v. Hence, $v \in V(\bigcup_{i=0}^{\infty} G_i)$.
Let $e = \{v_m, v_n\} \in E(G)$. Either e is contained in G_0 or P_i for some i. If not, there is a stage of the construction, such that v_m and v_n are already in G_i. Then e gets added before any other new vertex is added. □

The rest of this section will be concerned with the following question: Let $k \in \mathbb{N}$ and G an infinite graph. Is G k-connected if and only if for all finite $V' \subseteq V(G)$, $|V'| > k$ there exists a finite k-connected subgraph $G' \subseteq G$ that contains V'?

One direction of the statement is trivial for any k: If there is a finite k-connected subgraph for any finite set of vertices, then G is k-connected by Theorem 2. For the other direction, we will see that the Conjecture holds for $k = 1, 2$, but fails for $k \geq 3$. We will begin with the proofs for small k and then present the counter-examples for bigger k.
Let $V' = \{v_1, \dots, v_n\}$.
For $k = 1$ G is connected. Hence, for any pair i, j with $i \neq j$ there exists a v_i-v_j path P_{ij} in G. Thus, $G' := \bigcup_{i \neq j} P_{ij}$ is connected and contains all of V.
For $k = 2$ let G_2 be a cycle of G containing v_1 and v_2. Now let G_i, $i < n$, be a finite 2-connected subgraph of G containing $v_1, \dots, v_i$. If v_{i+1} is contained in G_i, then let $G_{i+1} = G_i$. Otherwise, apply Lemma 1 to obtain a finite 2-connected subgraph H of G containing v_{i+1}. Since H is a supergraph of G_i, it still contains $v_1, \dots, v_i$ and we can let $G_{i+1} = H$.

Theorem 5. *Let $k \geq 3$. There is an infinite k-connected graph G_k such that G_k is k-connected, but it has no finite k-connected subgraph.*

Proof. The graph G_k consists of $k-1$ *levels*, where each levels contains infinitely many complete graphs of order $k-1$. The complete graphs are grouped into (vertical) *slices* such that a slice contains exactly one complete graph at each level. The vertices of the complete graphs of one level lie all on $k-1$ double rays and the graphs of a single slice are connected as well. Figure 4 shows a part of G_5.

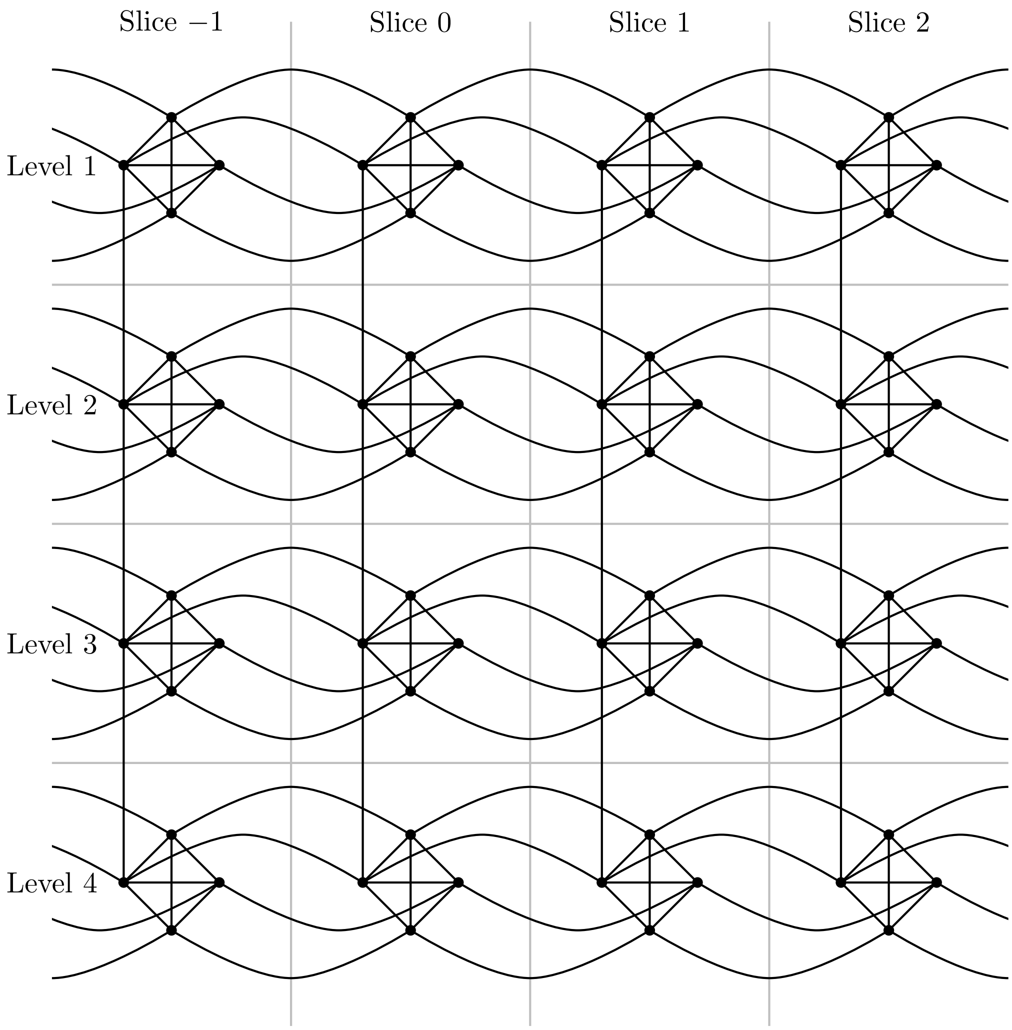

Figure 4: Part of G_5

To simplify notation, let $[n] = \{1, \dots n\}$ for $n \geq 1$. Now, define $G_k = (V_k, E_k)$ where $V_k = [k-1] \times [k-1] \times \mathbb{Z}$. Here, vertex (i, j, l) is the i-th vertex of a K_{k-1} on the j-th level and slice l. E_k consists of the following edges:

- $(i,j,l)(i',j,l)$ for $i,i',j \in [k-1]$, $i \neq i', l \in \mathbb{Z}$ that constitute the complete graphs,
- $(i,j,l)(i,j,l+1)$ for $i,j \in [k-1]$, $l \in \mathbb{Z}$ the *horizontal* double rays connecting the nodes on the levels, and
- $(1,j,l)(1,j+1,l)$ $i \in [k-1]$, $j \in [k-2]$, $l \in \mathbb{Z}$ connecting the different levels.

Note that all i-th vertices (of a single level) lie on the same double ray and two neighbouring levels are connected at every first vertex of the K_{k-1}.
To prove k-connectedness, we will construct k independent paths between any two vertices. Then, G_k is k-connected by Theorem 2. So, let $v_1 = (i_1, j_1, l_1)$ and $v_2 = (i_2, j_2, l_2)$. We will distinguish several cases depending on the relative positions of the vertices.
First, assume that they are in the same slice and the same level, i.e. $l_1 = l_2$ and $j_1 = j_2$. Then, there are $k-2$ idependent paths in the complete graph, to which the vertices belong to: v_1v_2 and $v_1(i, j_1, l_1)v_2$ for $i \in [k-1]\backslash\{i_1, i_2\}$. The remaining two paths take a detour via the neighbouring slices staying on the same level: $v_1(i_1, j_1, l_1-1)(i_2, j_1, l_1-1)v_2$ and $v_1(i_1, j_1, l_1+1)(i_2, j_1, l_1+1)v_2$.

Now, assume that v_1 and v_2 are still in the same slice, but on different levels, i.e. $l_1 = l_2$ but $j_1 \neq j_2$, say $j_1 < j_2$. Then, we have k paths going to different slices, going down some levels and then back to the original slice. Formally, we take the paths $v_1(i_1, j_1, l_1-1)(1, j_1, l_1-1)(1, j_1+1, l_1-1)\dots(1, j_2, l_1-1)(i_2, j_2, l_1-1)v_2$ going one slice to the left, $v_1(i, j_1, l_1)(i, j_1, l_1+1)\dots(i, j_1, l_1+i)(1, j_1, l_1+i)(1, j_1+1, l_1+i)\dots(1, j_2, l_1+i)(i, j_2, l_1+i)(i, j_2, l_1+i-1)\dots(i, j_2, l_1)v_2$ (for $i \in [k-1]\backslash\{i_1\}$) going i slices to the right, and $v_1(i_1, j_1, l_1+1)\dots(i, j_1, l_1+k)(1, j_1, l_1+k)(1, j_1+1, l_1+k)\dots(1, j_2, l_1+k)(i, j_2, l_1+k)(i, j_2, l_1+k-1)\dots(i, j_2, l_1)v_2$ going k slices to the right.
For the last two cases, we have $l_1 \neq l_2$ (say $l_1 < l_2$), i.e. the vertices are on different slices. We begin with the slightly easier case $j_1 = j_2$ with both vertices being on the same level. Then we have $k-1$ independent paths on the same level connecting the vertices: $v_1(i_1, j_1, l_1+1)\dots(i_1, j_1, l_2)v_2$ and $v_1(i, j_1, l_1)(i, j_1, l_1+1)\dots(i, j_1, l_2)(i_2, j_1, l_2)v_2$ for $i \in [k-1]\backslash\{i_1\}$. Furthermore, we take the path $v_1(i_1, j_1, l_1-1)(1, j_1, l_1-1)(1, j_1+1, l_1-1)(1, j_1+$

$1, l_1) \ldots (1, j_1+1, l_2+1)(1, j_1, l_2+1)(i_2, j_1, l_2+1)v_2$ which avoids the other paths by going down one level. If the vertices are on the lowest level, i. e. $j_1 = j_2 = k-1$, then go up one level by replacing j_1+1 by j_1-1.
In the last case, we have $l_1 \neq l_2$ (say $l_1 < l_2$) and $j_1 \neq j_2$ (say $j_1 < j_2$). Our first k_1 path will go to the left, down and then to v_2 as seen above. Formally, we take $v_1(i, j_1, l_1)(i, j_1, l_1-1) \ldots (i, j_1, l_1-i)(1, j_1, l_1-i)(1, j_1+1, l_1-i) \ldots (1, j_2, l_1-i)(i, j_2, l_1-i)(i, j_2, l_1-i+1) \ldots (i, j_2, l_2)v_2$ (for $i \in [k-1] \setminus \{i_1\}$) going i slices to the left starting at v_1, and $v_1(i_1, j_1, l_1-1) \ldots (i, j_1, l_1-k)(1, j_1, l_1-k)(1, j_1+1, l_1-k) \ldots (1, j_2, l_1-k)(i, j_2, l_1-k)(i, j_2, l_1-k+1) \ldots (i, j_2, l_2)v_2$ going k slices to the left. Our last path avoids all these paths by going to the right beyond v_2 and then down, i. e. we take $v_1(i_1, j_1, l_1+1) \ldots (i, j_1, l_2+1)(1, j_1, l_2+1)(1, j_1+1, l_2+1) \ldots (1, j_2, l_2+1)(i_2, j_2, l_2+1)v_2$. This proves the k-connectedness of G_k.
To complete the proof assume that there is a finite k-connected subgraph G of G_k. Let l be the leftmost slice of G, i. e. $V(G)$ does not contain a vertex (i, j, k) such that $k < l$, but a vertex $v = (i, k, l)$ and let $V_l \subseteq V(G)$ be the set of all vertices in that slice of G. If there exists a $v = (i, j, l) \in V_l$ such that $i > 1$, then v has degree less than k in G, since its edge to slice $l-1$ is not in G and it has degree k in G_k. Otherwise, all vertices in V_l are of the form $(1, j, l)$. Pick $v \in V_l$ that is in the smallest level, i. e. it has the minimal j value of all $v \in V_l$. Since it has no neighbours from its complete graph left, no edge to slice $l-1$ and no edge to the slice above it (or it is on slice 0), its degree in G is at most 2.
Thus, in both cases we can isolate a vertex of G by removing at most k vertices. So, G is not k-connected. □

This result gives for every $k \geq 3$ an infinite graph that is k-connected, but has no finite k-connected subgraphs. This raises the question whether even higher connectivity ensures the existence of k-connected finite subgraphs. Formally, is there for every k an $l > k$ such that every infinite l-connected graphs has finite k-connected subgraphs? But even this does not hold:

Theorem 6. *There is an infinitely connected graph G that has no finite 3-conected subgraphs.*

Proof. We construct G inductively starting with a single edge. At stage $i+1$ we add for any pair of vertices x, y a new vertex and connect it both x and y. The resulting graphs is obviously infinitely connected, since it contains infinitely many paths of length two between any pair of vertices. Now assume G has a finite 3-connected subgraph G'. There is a minimal stage $i > 0$ such that all vertices of G' are added to G in that stage. Consider a vertex of G' that was added in stage i. It has at most two neighbours in G', since all other neighbours are added in stages greater than i. Thus G' is not 3-connected. □

Since G has no finite 3-connected subgraphs, it has no finite k-connected subgraphs for any $k > 2$, which answers our question from above. Obviously, this result implies Theorem 5.

References

[1] R. Diestel. *Graph Theory.* Springer, third edition edition, 2006.