On the independence number of graphs with maximum degree 3

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Abstract

Let G be an undirected graph with maximum degree at most 3 such that G does not contain any of the three graphs shown in Figure 1 as a subgraph. We prove that the independence number of G is at least $n(G)/3 + nt(G)/42$, where $n(G)$ is the number of vertices in G and $nt(G)$ is the number of nontriangle vertices in G. This bound is tight as implied by the well-known tight lower bound of $5n(G)/14$ on the independence number of triangle-free graphs of maximum degree at most 3.

We show an algorithmic application of the aforementioned combinatorial result to the area of parameterized complexity. We present a linear-time kernelization algorithm for the independent set problem on graphs with maximum degree at most 3 that computes a kernel of size at most $420k/141 < 3k$, where k is the given parameter. This improves the known $3k$ upper bound on the kernel size for the problem.

1 Introduction

We study the independence number of graphs with maximum degree at most 3. This study is motivated by the importance of the independent set problem on graphs with maximum degree at most 3, abbreviated IS-3: Given an undirected graph G with maximum degree at most 3 and a nonnegative integer k, decide if G has an independent set of cardinality at least k. This problem is known to be NP-complete [9], and the optimization version of the problem has received significant interest from both areas of approximation and exact algorithms. After a long sequence of results in each area, up to the authors' knowledge, the currently-best approximation algorithm for the problem achieves a ratio that is arbitrarily close to 6/5 [1], and the currently-best exact algorithm runs in time $O(1.0885^{n(G)})$, where $n(G)$ is the number of vertices in G [15].

We take a combinatorial approach to the problem, establishing lower bounds on the independence number of a graph with maximum degree at most 3 that excludes the three obstacle-graphs depicted in Figure 1 as subgraphs. Combinatorial results of a similar nature are very common in the literature. Brook's theorem [4], published as early as 1941, implies that the independence number of a K_4-free graph G with maximum degree at most 3 is at least $n(G)/3$, where $n(G)$ is the number of vertices in G. Staton showed in 1979 [13] that the independence number of a triangle-free graph G with maximum degree at most 3 is at least $5n(G)/14$. Staton's lower bound for triangle-free graphs is tight, as shown by the example given in [7]. A simpler proof of Staton's result was given by Jones in 1990 [12], and an even simpler proof was given by Heckman and Thomas in 2001 [11]. In their result [11], Heckman and Thomas define the notion of a *difficult component* in a graph, based on some "obstacle" subgraphs. They then prove that every triangle-free graph with maximum degree

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at most 3 has an independence number of at least $(4n(G) - e(G) - \lambda(G))/7$, where $e(G)$ and $\lambda(G)$ are the number of edges and the number of difficult components in G, respectively. They showed how their result implies Staton's result [13]. We mention that for a *connected* triangle-free graph with maximum degree at most 3, Fraughnaugh and Locke proved a lower bound of $(11n(G)-4)/30$ on its independence number, which is strictly larger than $5n(G)/14$ for $n(G) \geq 15$ [8].

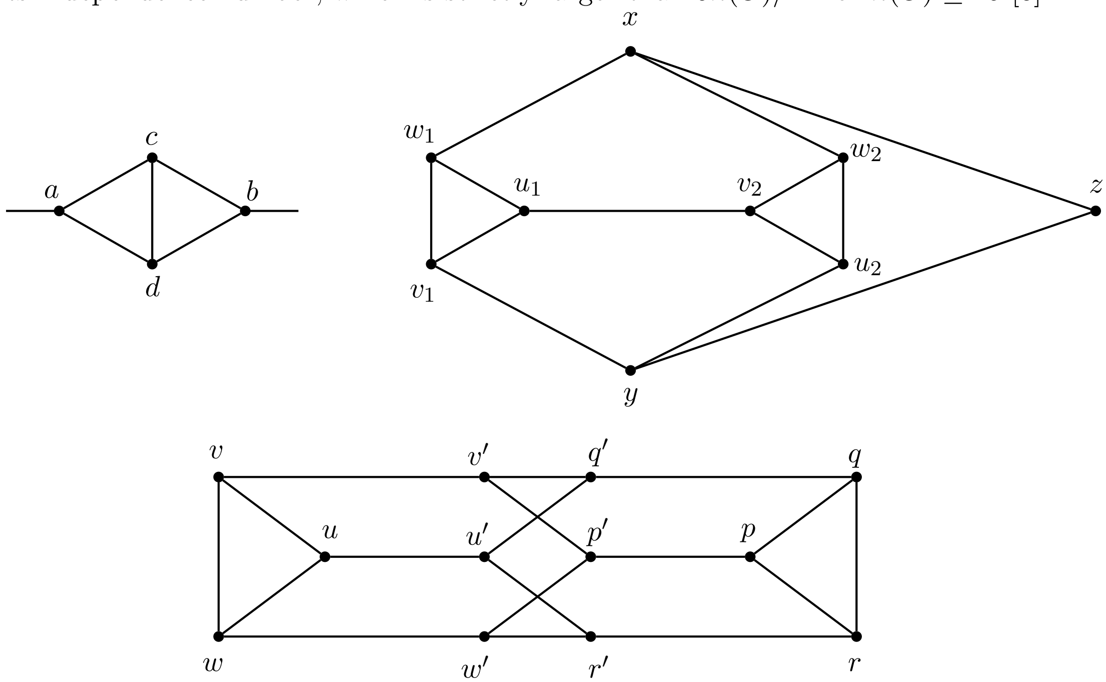

Figure 1: The obstacle graphs. The graph on the top-left is referred to as the *small obstacle*; the graph on the top-right is referred to as the *medium obstacle*; and the graph at the bottom is referred to as the *large obstacle*. The degree of vertex z in the medium obstacle could be either 2 or 3, and the degree of vertices a and b in the small obstacle is 3, and a and b could be adjacent.

Very recently (2008), Harant et al. [10] generalized Heckman and Thomas' result to graphs of maximum degree at most 3 that may contain triangles. They define the notion of a *difficult block*, which is a block (a biconnected component) that is isomorphic to one of the four obstacle graphs given in Figure 9. They use the notion of difficult blocks to define the *bad components* of a graph: the components in which every block is either a difficult block or an edge between two difficult blocks. They then prove that the independence number of a K_4-free graph with maximum degree at most 3 is at least $(4n(G) - e(G) - \lambda(G) - tr(G))/7$, where $\lambda(G)$ and $tr(G)$ are the number of bad components and the number of vertex-disjoint triangles in G, respectively.

The lower bound given by Harant et al. in [10] may become meaningless when the number of triangles in G is large. For example, consider the two K_4-free 3-regular graphs depicted in Figure 5 in Section 7. The number of (vertex-disjoint) triangles in the graph on the left of the figure is $n(G)/4$, and the graph contains no bad components. Therefore, we have $tr(G) = n(G)/4$, $\lambda(G) = 0$, and $e(G) = 3n(G)/2$. In this case the result in [10] gives a lower bound of $9n(G)/28$ on the independent number of G, which is even worse than the $n(G)/3$ lower bound given by Brook's theorem [4]. Similarly, the result in [10] gives a lower bound of $n(G)/3$ on the independence number of the graph on the right of Figure 5, matching the lower bound given by Brook's theorem [4]. Note that the independence number of the graph on the left of Figure 5 is $3n(G)/8$, and that of the graph on the right of Figure 5 is $5n(G)/12$.

Since at most one vertex from a triangle can be in any maximum independent set of G, the presence of a lot of triangles poses an immediate obstacle for obtaining a lower bound on the independence number that is larger than $n(G)/3$. Intuitively, one would think that we should be able to "gain" a certain fraction of the number of nontriangle vertices in G (i.e., vertices that do not appear in any triangle), above the "guaranteed" $n(G)/3$ lower bound on the independence number. This impression, however, is incorrect, as can be seen from the middle and last graphs in Figure 1: the independence number of each of these two graphs is exactly $n(G)/3$, despite the presence of nontriangle vertices in it. A natural question to ask then is whether there are certain "obstacle" subgraphs that can be excluded from G, so that a lower bound of the form $n(G)/3 + nt(G)/c$ on the independence number can be derived, where $nt(G)$ is the number of nontriangle vertices in G, and c is some fixed (proper) constant; such a result can be interpreted as we are gaining a fraction $n(G)/c$ of the nontriangle vertices in G above the guaranteed value of $n(G)/3$.

In the current paper we prove the following combinatorial result: if G is a graph with maximum degree at most 3 that does not contain any of the three obstacle graphs depicted in Figure 1 as a subgraph, then the independence number of G is at least $n(G)/3 + nt(G)/42$. This lower bound on the independence number in terms of the number of vertices and the number of nontriangle vertices in the graph is tight, as implied by the well-known tight lower bound of $5n(G)/14 = n(G)/3 + nt(G)/42$ on the independence number of triangle-free graphs of maximum degree at most 3.[1] The technique employed in proving the aforementioned result is the following. We apply a sequence of operations to G in three phases. In the first phase, we apply operations to G to obtain a graph G_1 in which each triangle is contained in a special structure that we call a *steeple*. None of these operations decreases the number of nontriangle vertices in the graph, and each of them guarantees that the independence number of the graph to which the operation is applied is at least as large as that of the resulting graph, plus one third the number of vertices removed by the operation. In the second phase, we apply operations to G_1 to simplify its structure further, and to make the steeples in the resulting graph G_2, and hence the triangles, farther apart. Each of these operations removes a subgraph H from G_1 satisfying the local ratio: an independent set of H of size at least $n(H)/3 + nt(H)/42$ can be added to any independent set of the resulting graph. Finally, a lower bound of $n(G_2)/3 + nt(G_2)/42$ is established on the independence number of G_2 by applying another sequence of operations to G_2, and using a charging argument coupled with an amortized analysis. This implies a lower bound of $n(G)/3+nt(G)/42$ on the independence number of G. Up to the authors' knowledge, the technique of using amortized analysis for establishing lower bounds on the independence number is new, and could turn out to have wider applications.

In addition to the tightness of combinatorial lower bound result established in the current paper, the simplicity of the lower bound expression, and the fact that the three obstacle graphs depicted in Figure 1 can be pre-processed (removed) in polynomial time, allows this result to have algorithmic applications. As shown in Section 5, by introducing some reduction rules that allow us to lower bound the value of $nt(G)$ by $n(G)/10$ in the resulting graph, and then using the combinatorial result in the current paper, we obtain a kernelization algorithm for the IS-3 problem that produces a kernel of size at most $420k/141$ in $O(k)$ time.[2] We note that since a K_4 subgraph must appear as a separate component in a graph of maximum degree 3, Brook's theorem [4] implies a kernel of size at most $3k$ for IS-3.

[1] It is easy easy to see that the coefficient of $n(G)$ in such an expression cannot be improved by considering a graph that is a chain of triangles.

[2] Note that a graph G with $n(G)/10$ nontriangle vertices could have $3n/10$ triangles, and Harant et al.'s result [10] would give a lower bound of only $11n(G)/35$ on its independence number (if G is 3-regular).

The $420k/141$ upper bound on the kernel size for IS-3 implies a lower bound of $420k/279$ on the kernel size for the vertex cover problem on graphs with maximum degree at most 3, abbreviated VC-3, as shown in Section 6. Both results are in line with the recent progress in deriving lower and upper bounds on the kernel size for certain fixed-parameter tractable problems [2, 3].

The paper is organized as follows. In Section 2 we give the necessary notations and terminologies used throughout the paper. In Section 3 we give some structural results that are essential for the later sections. The main combinatorial result is given in Section 4. The kernelization algorithm is given in Section 5, and the lower bound result on the kernel size for VC-3 is given in Section 6.

2 Preliminaries

We assume familiarity with the basic notations and terminologies about graphs. For more information, we refer the reader to West [14]. We only consider simple undirected graphs in this paper.

For a graph G we denote by $V(G)$ and $E(G)$ the set of vertices and edges of G, respectively; $n(G) = |V(G)|$ and $e(G) = |E(G)|$ are the number of vertices and edges in G. A set of vertices in $V(G)$ is said to be an *independent set* if no edge in $E(G)$ exists between any two vertices in this set. By $\alpha(G)$ we denote the *independence number* of G, that is, the size of a maximum independent set in G.

For a set of vertices $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by the set of vertices in S. For a vertex $v \in G$, $G - v$ denotes $G[V(G) \setminus \{v\}]$, and for a subset of vertices $S \subseteq V(G)$, $G - S$ denotes $G[V(G) \setminus S]$. By *removing* a subgraph H of G we mean removing $V(H)$ from G to obtain $G - V(H)$. For two vertices $u, v \in V(G)$, we denote by $G - (u, v)$ the graph $(V(G), E(G) \setminus \{(u, v)\})$, and by $G + (u, v)$ the *simple* graph $(V(G), E(G) \cup \{(u, v)\})$.

The *degree* of a vertex v in G, denoted $d(v)$, is the number of edges in G that are incident to v. The *degree* of G, denoted $\Delta(G)$, is $\Delta(G) = \max\{d(v) \mid v \in G\}$.

Call a vertex $v \in G$ a *triangle vertex* if v is a vertex of some triangle in G; otherwise call v a *nontriangle vertex*. We denote the number of vertex-disjoint triangles in G by $tr(G)$, and the number of nontriangle vertices in G by $nt(G)$.

Two triangles in a graph G are said to *share an edge* if the two triangles have exactly two vertices in common. Two triangles are said to be *adjacent* if the two triangles do not have any common vertex and a vertex in one of the triangles is adjacent to a vertex in the other triangle. Note that if a graph has maximum degree at most 3, then no two triangles in the graph can have exactly one vertex in common.

The *blocks* of a graph G are its maximal 2-connected subgraphs, its cut-edges, and its isolated vertices. Two blocks may only intersect at a cut-vertex of G. The *block-cutpoint tree* of a connected graph G is the tree whose vertices are the blocks and cut-vertices of G, with an edge from cut-vertex to each block that contains it. A connected graph that is not 2-connected has a nontrivial block-cutpoint tree; its *leaf blocks* are its blocks that are leaves in its block-cutpoint tree.

A *parameterized problem* is a set of instances of the form (x, k), where $x \in \Sigma^*$ for a finite alphabet set Σ, and k is a non-negative integer called the *parameter* [6]. A parameterized problem Q is *kernelizable* [6] if there exists a polynomial-time computable reduction that maps an instance (x, k) of Q to another instance (x', k') of Q such that: (1) $|x'| \leq g(k)$ for some recursive function g, (2) $k' \leq k$, and (3) (x, k) is a yes-instance of Q if and only if (x', k') is a yes-instance of Q. The instance x' is called the *kernel* of x. For more information on parameterized complexity and kernelization we refer the reader to [6].

The INDEPENDENT SET problem on graphs of maximum degree at most 3, abbreviated IS-3, is defined as follows:

> IS-3. Given an undirected graph G with $\Delta(G) \leq 3$, and a nonnegative integer k, determine if G has an independent set of size at least k.

3 Structural results

We present in this section some structural results that will be used in the remaining sections of the paper. Some of these results are also of independent interest.

Let G be a graph such that $\Delta(G) \leq 3$. The following two facts can be easily verified by the reader:

Fact 3.1 *Let (u, v, w) be a triangle in G such that $d(u) = 2$. Then there exists a maximum independent set of G that contains u.*

Fact 3.2 *Let (u, v, w) and (p, v, w) be two triangles in G that share an edge (v, w). Then there exists a maximum independent set of G that excludes v (or w).*

We assume in the remaining discussion in this section that no triangle in G contains a vertex of degree 2, and that no two triangles in G share an edge. Therefore, since no two (distinct) triangles in G can share exactly one vertex, any two triangles in G are vertex-disjoint.

A sequence of distinct triangles $T_1, \ldots, T_\ell$, $\ell \geq 1$, in G is said to form a *path of triangles* if either $\ell = 1$, or if $\ell > 1$ and triangle T_i is adjacent to T_{i+1}, for $i = 1, \ldots, \ell - 1$. A path of triangles $T_1, \ldots, T_\ell$ is said to be a *cycle of triangles* if either $\ell > 2$ and T_1 and T_ℓ are adjacent, or $\ell = 2$ and (some) two vertices of T_1 are neighbors of two vertices of T_2 (i.e., there are at least two edges between the vertices of T_1 and the vertices of T_2). The *length* of a path/cycle of triangles is the number of triangles in it. A path of triangles is *maximal* if it is maximal under containment.

Lemma 3.3 *Let $T_1, \ldots, T_\ell$ be a cycle of triangles in G, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, u_i is adjacent to v_{i+1} for $i = 1, \ldots, \ell - 1$, and u_ℓ is adjacent to v_1. Then there exists a maximum independent set of G that contains $\{v_1, \ldots, v_\ell\}$.*

PROOF. Observe first that all the neighbors of the vertices $\{v_1, \ldots, v_\ell\}$ are vertices from triangles $T_1, \ldots, T_\ell$. If I_{max} is a maximum independent set of G, then I_{max} contains at most one vertex from each of triangles $T_1, \ldots, T_\ell$, and hence I_{max} contains at most ℓ vertices from triangles $T_1, \ldots, T_\ell$. It follows from the previous statements that if we replace the vertices in $I_{max} \cap (\bigcup_{i=1}^{\ell} V(T_i))$ with $\{v_1, \ldots, v_\ell\}$, we obtain a maximum independent set of G that contains $\{v_1, \ldots, v_\ell\}$. $\square$

Lemma 3.4 *Let $T_1 = (u_1, v_1, w_2)$ and $T_2 = (u_2, v_2, w_2)$ be two adjacent triangles in G where u_1 is adjacent to v_2. Suppose that w_1 and w_2 share a common neighbor x, v_1 and u_2 share a common neighbor y, and x and y share a common neighbor z; that is, the subgraph of G induced by $V(T_1) \cup V(T_2) \cup \{x, y, z\}$ is a medium obstacle graph (see Figure 1). Then there exists a maximum independent set of G containing the set of vertices $\{x, y, v_2\}$.*

Proof. Let I_{max} be a maximum independent set of G.

If I_{max} contains z, then I_{max} excludes both x and y, and by maximality, I_{max} contains exactly one vertex from each of triangles T_1, T_2. Therefore, we can replace z and the 2 vertices from T_1, T_2 in I_{max} with $\{x, y, v_2\}$ to obtain an independent set of G of the same cardinality as I_{max}, and hence this independent set is a maximum independent set of G.

If I_{max} excludes z, then since the maximum independent set of the subgraph of G induced by the set of vertices $V(T_1) \cup V(T_2) \cup \{x, y\}$ has size 3, I_{max} contains exactly 3 vertices from $V(T_1) \cup V(T_2) \cup \{x, y\}$. Those 3 vertices can be replaced with the vertices in $\{x, y, v_2\}$, to obtain a maximum independent set of G containing vertices x, y, v_2.

It follows that there exists a maximum independent set of G that contains the set of vertices $\{x, y, v_2\}$. This completes the proof. $\square$

4 A combinatorial result

Let G be a graph with $\Delta(G) \leq 3$ such that G does not contain any of the three graphs depicted in Figure 1 as a subgraph. These three graphs present an obstacle for the combinatorial lower bound that we derive on the independence number of G. We call the three graphs depicted in Figure 1 the *obstacle graphs*. The graph on the top-left of Figure 1 is referred to as the *small obstacle*, the graph on the top-right of Figure 1 is referred to as the *medium obstacle*, and the graph on the bottom is referred to as the *large obstacle*. The degree of each of the two vertices a and b in the small obstacle is 3 and a and b could be adjacent, and the degree of vertex z in the medium obstacle could be either 2 or 3. Note that since every vertex in the large obstacle has degree 3, if the large obstacle appears in G then it must appear as a separate component. Note also that since G does not contain a small obstacle as a subgraph, G is K_4-free; and that G contains the medium obstacle as an induced subgraph if and only if it contains it as a subgraph. We will say that a graph is *obstacle-free* if the graph does not contain any of the three obstacle graphs depicted in Figure 1 as a subgraph.

This section is devoted to proving that $\alpha(G) \geq n(G)/3 + nt(G)/42$. The proof proceeds in three phases. In the first phase, we apply a set of graph operations to G to obtain a "simplified" graph. The operations performed in the first phase reduce the number of triangles in G without affecting the nontriangle vertices; these operations also ensure that given any independent set I' of the resulting graph, an independent set I of G containing I' can be obtained by adding a vertex from each of the triangles removed by one of these operations.

Let G_1 be the graph resulting from G at the end of the first phase. In the second phase we apply more operations to G_1 to simplify its structure further. In contrast to the operations performed in the first phase, the operations performed in the second phase may remove nontriangle vertices. Each of these operations removes a subgraph H from G_1 to obtain a subgraph G_1' (i.e., $G_1' = G_1 - V(H)$) such that there exists a subset of vertices $S_H \subseteq V(H)$ that is an independent set satisfying: (1) $\alpha(G_1) \geq |S_H| + \alpha(G_1')$, (2) $|S_H| \geq n(H)/3 + nt(H)/42$, and (3) $nt(G_1) = nt(H) + nt(G_1')$.

Let G_2 be the graph resulting from G_1 at the end of the second phase. In the third phase we prove that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$. To do so, we prove using an amortized analysis technique that $\alpha(G_2) \geq (23n(G_2) - 6e(G_2) + nt(G_2))/42$. Since $e(G_2)$ is at most $3n(G_2)/2$, the desired statement follows.

It is essential that none of the operations used in this section when applied to a graph that is obstacle-free produces an obstacle subgraph in the resulting graph, or produces a graph with maximum degree larger than 3. The following observation will be useful in proving the previous statements:

Observation 4.1 *Let G be a graph with $\Delta(G) \leq 3$ such that G is obstacle-free. Then for any subset of vertices $S \subseteq V(G)$, the subgraph $G - S$ of G has maximum degree at most 3 and is obstacle-free.*

PROOF. It is clear that $G-S$ has maximum degree at most 3. The fact that $G-S$ is obstacle-free follows from the fact that $\Delta(G) \leq 3$, and that every vertex in an obstacle, except z, a, b which could have degree 2 or 3, has degree 3. Therefore, if an obstacle graph did no already exist in G then the removal of a subset of vertices from G will not create an obstacle graph. $\square$

4.1 The first phase

In what follows we introduce a set of graph operations to be applied to the graph G to obtain a simpler graph in which every triangle is contained in one of two specific structures. We will need to keep track of how each operation affects the number of vertices, the number of nontriangle vertices, and the independence number of the graph G. For convenience, if an operation, or a set of operations, is applied to G to obtain a graph G', we will denote by $\delta_{n(G)}$, $\delta_{nt(G)}$, and $\delta_{\alpha(G)}$ the entities $n(G) - n(G')$, $nt(G) - nt(G')$, and $\alpha(G) - \alpha(G')$, respectively. Each of the operations that follow is justified by the lemma preceding it.

Lemma 4.2 *Let (u, v, w) be a triangle in G such that one of its vertices is of degree 2. Let $G' = G - \{u, v, w\}$. Then $\delta_{n(G)} = 3$, $\delta_{\alpha(G)} = 1$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

PROOF. Since G' is obtained from G by removing 3 vertices, we have $\delta_{n(G)} = 3$. The fact that $\delta_{\alpha(G)} = 1$ follows from Fact 3.1, and since u, v, w are triangle vertices, we have $\delta_{nt(G)} \leq 0$. Finally, by Observation 4.1, $\Delta(G') \leq 3$ and G' is obstacle-free. $\square$

Operation 4.1 *Let (u, v, w) be a triangle in G such that one of its vertices is of degree 2. Then set $G := G - \{u, v, w\}$.*

Lemma 4.3 *Let (u, v, w) be a triangle in G such that $d(u) = d(v) = d(w) = 3$. Let u', v', and w' be the neighbors of u, v, w, respectively that are nontriangle vertices. Suppose that two vertices in $\{u', v', w'\}$ are adjacent, and let $G' = G - \{u, v, w\}$. Then $\delta_{n(G)} = 3$, $\delta_{\alpha(G)} \geq 1$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

PROOF. It is clear that $\delta_{n(G)} = 3$. Since two vertices in $\{u', v', w'\}$ are adjacent, say vertices u' and v', any maximum independent set I' of G' contains at most one vertex from $\{u', v'\}$. If $u' \notin I'$, then $I' \cup \{u\}$ is an independent set of G. On the other hand, if $v' \notin I'$, then $I' \cup \{v\}$ is an independent set of G. It follows that $\delta_{\alpha(G)} \geq 1$. Since the removed vertices u, v, w are all triangle vertices, we have $\delta_{nt(G)} \leq 0$. By Observation 4.1, $\Delta(G') \leq 3$ and G' is obstacle-free. $\square$

Operation 4.2 *Let (u, v, w) be a triangle in G such that $d(u) = d(v) = d(w) = 3$. Let u', v', and w' be the neighbors of u, v, w, respectively that are nontriangle vertices. If two vertices in $\{u', v', w'\}$ are adjacent then set $G := G - \{u, v, w\}$.*

Lemma 4.4 *Let $T_1, \ldots, T_\ell$ be a cycle of triangles, where $T_i = (u_i, v_i, w_i)$, $i = 1, \ldots, \ell$, u_i is adjacent to v_{i+1} for $i = 1, \ldots, \ell - 1$, and u_ℓ is adjacent to v_1. Let G' be the subgraph of G obtained by removing the vertices in the triangles $T_1, \ldots, T_\ell$ (i.e., $G' = G - \bigcup_{i=1}^{\ell} V(T_i)$). Then $\delta_{n(G)} = 3\ell$, $\delta_{\alpha(G)} \geq \ell$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

PROOF. Since G' is obtained from G by removing exactly 3ℓ vertices, we have $\delta_{n(G)} = 3\ell$. The fact that $\delta_{\alpha(G)} \geq \ell$ follows from Lemma 3.3. Since none of the vertices removed is a nontriangle vertex, we have $\delta_{nt(G)} \leq 0$. The facts that $\Delta(G') \leq 3$ and that G' is obstacle-free follow from Observation 4.1. □

Operation 4.3 *Let $T_1, \ldots, T_\ell$ be a cycle of triangles, where $T_i = (u_i, v_i, w_i)$, $i = 1, \ldots, \ell$, u_i is adjacent to v_{i+1} for $i = 1, \ldots, \ell - 1$, and u_ℓ is adjacent to v_1. Then set $G := G - \bigcup_{i=1}^{\ell} V(T_i)$.*

Lemma 4.5 *Let $T_1, \ldots, T_\ell$, $\ell > 2$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell - 1$. Suppose that w_1 and w_ℓ share a common neighbor x, v_1 and u_ℓ share a common neighbor y, and x and y share a common neighbor z. Let G' be the graph resulting from G after removing the set of vertices of triangles $T_1, T_3, \ldots, T_\ell$ and adding the two edges (x, v_2) and (y, u_2); that is, $G' = (G - V(T_1) - \bigcup_{i=3}^{\ell} V(T_i)) + (x, v_2) + (y, u_2)$. Then $\delta_{n(G)} = 3(\ell - 1)$, $\delta_{\alpha(G)} \geq \ell - 1$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

PROOF. The fact that $\delta_{n(G)} = 3(\ell - 1)$ follows from the fact that $|V(G)| = |V(G')| + |V(T_1)| + |\bigcup_{i=3}^{\ell} V(T_i)|$, and that $|V(T_1)| + |\bigcup_{i=3}^{\ell} V(T_i)| = 3(\ell - 1)$.

Now to show that $\delta_{\alpha(G)} \geq \ell - 1$, let I' be a maximum independent set of G'. If I' contains both x and y, then I' must exclude v_2 and u_2. In this case $I = I' \cup \{u_1\} \cup \{v_3, \ldots, v_\ell\}$ is an independent set in G of size $|I'| + (\ell - 1)$. If I' excludes x, then $I = I' \cup \{w_1, w_\ell\} \cup \{u_3, \ldots, u_{\ell-1}\}$ if $\ell > 3$ and $I = I' \cup \{w_1, w_3\}$ if $\ell = 3$, is an independent set in G of size $|I'| + (\ell - 1)$. If I' excludes y, then $I = I' \cup \{v_1\} \cup \{u_3, \ldots, u_\ell\}$ is an independent set in G of size $|I'| + (\ell - 1)$.

It follows that G has an independent set of size $\alpha(G') + \ell - 1$, and hence, $\delta_{\alpha(G)} \geq \ell - 1$.

Due to the addition of edges (x, v_1) and (y, u_2), the only vertices in G' whose degrees could have increased are vertices x, v_2, y, u_2. However, at least one neighbor of each of these vertices was removed from G; therefore, the degree of each of those vertices is at most 3 in G'. It follows that $\Delta(G') \leq 3$. Now since all vertices removed from G are triangle vertices, to show that $\delta_{nt(G)} \leq 0$, it suffices to show that the addition of the two edges (x, v_2) and (y, u_2) does not create any triangles. The addition of these edges can create a triangle only if x or y is a neighbor in G of a vertex of $V(T_2)$. The neighbors of x in G are w_1, w_ℓ, z and those of y are v_1, u_ℓ, z. Since all the triangles are vertex-disjoint, and since z is a nontriangle vertex (z is adjacent to both x and y), neither x nor y can be a neighbor in G of a vertex in $V(T_2)$. To show that G' is obstacle-free, similar to the above, it suffices to show that the addition of edges (x, v_2) and (y, u_2) does not create obstacles. It is easy to see that the addition of these edge cannot create a small obstacle because each of these two edges has one endpoint that is a nontriangle vertex in G' (x and y). Now each of x and y is a degree-2 vertex in G', and is adjacent to exactly one triangle vertex in G' (since z is a nontriangle vertex in G'). Therefore, neither x nor y can be a vertex of medium or a large obstacle in G', and hence the addition of the two edges (x, v_1) and (y, u_2) does not create obstacle graphs in G'. □

Operation 4.4 *Let $T_1, \ldots, T_\ell$, $\ell > 2$ be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell - 1$. If w_1 and w_ℓ share a common* *neighbor x, v_1 and u_ℓ share a common neighbor y, and x and y share a common neighbor z, then set $G := (G - (V(T_1) \cup \bigcup_{i=3}^{\ell} V(T_i))) + (x, v_2) + (y, u_2)$.*

Lemma 4.6 *Let $T_1, \ldots, T_\ell$, $\ell > 1$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell-1$. Suppose that w_1 and w_ℓ share a common neighbor x and v_1 and u_ℓ share a common neighbor y, and x and y do not share a neighbor. Let G' be the graph resulting from G by removing the set of vertices $\bigcup_{i=1}^{\ell} V(T_i)$ and adding the edge (x, y) (if (x, y) is not already an edge); that is $G' = (G - \bigcup_{i=1}^{\ell} V(T_i)) + (x, y)$. Then $\delta_{n(G)} = 3\ell$, $\delta_{\alpha(G)} \geq \ell$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

Proof. The fact that $\delta_{n(G)} = 3\ell$ follows from the fact that $|V(G)| = |V(G')| + |\bigcup_{i=1}^{\ell} V(T_i)|$, and that $\bigcup_{i=1}^{\ell} V(T_i)$ contains precisely 3ℓ vertices. To show that $\delta_{\alpha(G)} \geq \ell$, let I' be a maximum independent set of G'. Since $(x, y) \in E(G')$, I' contains at most one vertex from $\{x, y\}$. If I' excludes x, then $I' \cup \{w_1\} \cup \{v_i : i = 2, \ldots, \ell\}$ is an independent set in G of size $|I'| + \ell = \alpha(G') + \ell$. On the other hand, if I' excludes y, then $I' \cup \{u_i : i = 1, \ldots, \ell\}$ is an independent set in G of size $|I'| + \ell = \alpha(G') + \ell$. It follows that G has an independent set of size $\alpha(G') + \ell$, and hence, $\delta_{\alpha(G)} \geq \ell$.

Since vertices x and y do not share a neighbor in G, the addition of edge (x, y) will not create a triangle. This, together with the fact that all vertices removed are triangle vertices, imply that $\delta_{nt(G)} \leq 0$. Now since both x and y are nontriangle vertices of degree at most 2 in G' (since two neighbors of each of x, y were removed), edge (x, y) cannot be an edge in an obstacle graph in G', and hence its addition does not create obstacle graphs. Moreover, due to the fact that each of x and y has degree at most 2 in G', we have $\Delta(G') \leq 3$. □

Operation 4.5 *Let $T_1, \ldots, T_\ell$, $\ell > 1$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell-1$. Suppose that w_1 and w_ℓ share a common neighbor x and v_1, u_ℓ share a common neighbor y, and x and y do not share a neighbor. Then set $G := (G - \bigcup_{i=1}^{\ell} V(T_i)) + (x, y)$.*

Lemma 4.7 *Suppose that every triangle vertex in G has degree 3, and let $T_1, \ldots, T_\ell$, $\ell > 1$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell - 1$. Suppose that there exists a vertex in T_ℓ that does not share a common neighbor with v_1 and does not share a common neighbor with w_1. Then there exists a vertex in T_ℓ, say w_ℓ, that does not share a common neighbor with v_1 and does not share a common neighbor, and such that the following is true. Let w'_ℓ be the nontriangle vertex that is a neighbor of w_ℓ. Let G' be the graph resulting from G by removing the set of vertices $\bigcup_{i=2}^{\ell} V(T_i)$ and adding the edge (w'_ℓ, u_1); that is, $G' = (G - \bigcup_{i=2}^{\ell} V(T_i)) + (w'_\ell, u_1)$. Then $\delta_{n(G)} = 3(\ell - 1)$, $\delta_{\alpha(G)} \geq \ell - 1$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

Proof. The fact that $\delta_{n(G)} = 3(\ell-1)$ follows from the fact that $|V(G)| = |V(G')| + |\bigcup_{i=2}^{\ell} V(T_i)|$, and that $\bigcup_{i=2}^{\ell} V(T_i)$ contains precisely $3(\ell - 1)$ vertices.

To show that $\delta_{\alpha(G)} \geq \ell - 1$, let I' be a maximum independent set of G'. Since $(w'_\ell, u_1) \in E(G')$, I' contains at most one vertex from $\{w'_\ell, u_1\}$. If I' excludes u_1, then $I' \cup \{v_i : i = 2, \ldots, \ell\}$ is an independent set in G of size $|I'| + \ell - 1 = \alpha(G') + \ell - 1$. On the other hand, if I' excludes w'_ℓ, then $I' \cup \{w_\ell\} \cup \{u_i : i = 2, \ldots, \ell - 1\}$ is an independent set in G of size $|I'| + \ell - 1 = \alpha(G') + \ell - 1$. It follows that G has an independent set of size $\alpha(G') + \ell - 1$, and hence, $\delta_{\alpha(G)} \geq \ell - 1$.

Now since w'_ℓ is a not a neighbor of w_1 nor of v_1, the addition of edge (w'_ℓ, u_1) does not create a triangle. Since all vertices removed from G are triangle vertices, we have $\delta_{nt(G)} \leq 0$. By maximality of the path of triangles $T_1, \ldots, T_\ell$, vertex w'_ℓ is a nontriangle vertex, and T_1 is not adjacent to any triangle in G'. Moreover, since T_1 is a triangle in a path of at least two triangles, T_1 does not share an edge with another triangle. It follows from the previous statements that edge (w'_ℓ, u_1) cannot be an edge in a small or a medium obstacle, and hence its addition does not create a small nor a medium obstacle graph. Now it is possible that the addition of edge (w'_ℓ, u_1) creates a large obstacle, which must be a separate component in the resulting graph. However, if this is the case, then vertex v_ℓ of T_ℓ does not share a neighbor with v_1, nor does it share a neighbor with w_1. In this case we replace w_ℓ with v_ℓ, that is, we let G' be the graph resulting from G by removing the set of vertices $\bigcup_{i=2}^{\ell} V(T_i)$ and adding the edge (v'_ℓ, u_1), where v'_ℓ is the nontriangle vertex that is a neighbor of v_ℓ; that is, $G' = (G - \bigcup_{i=2}^{\ell} V(T_i)) + (v'_\ell, u_1)$. It is easy to see now that the modified operation cannot create any of the three obstacle graphs (given that the original operation creates a large obstacle). Finally, since at least one neighbor of each of w'_ℓ and u_1 was removed from G, we have $\Delta(G') \leq 3$. □

Operation 4.6 *Let $T_1, \ldots, T_\ell$, $\ell > 1$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell - 1$. Suppose that a vertex in T_ℓ, say w_ℓ, does not share a common neighbor with v_1 and does not share a common neighbor with w_1. Let w'_ℓ be the nontriangle vertex that is a neighbor of w_ℓ. Then set $G := (G - \bigcup_{i=2}^{\ell} V(T_i)) + (w'_\ell, u_1)$.*

Lemma 4.8 *Suppose that no two triangles in G are adjacent. Let (u, v, w) be a triangle in G such that $d(u) = d(v) = d(w) = 3$. Let u', v', and w' be the neighbors of u, v, w, respectively, that are nontriangle vertices, and assume that no edge exists between any two vertices of $\{u', v', w'\}$ (i.e., the subgraph of G induced by $\{u', v', w'\}$ is an independent set). Suppose further that there are two vertices in $\{u', v', w'\}$ that do not share a common neighbor in G. Then there are two vertices in $\{u', v', w'\}$, say u' and v', that do not share a common neighbor in G and such that the following is true. Let G' be the graph resulting from G by removing the set of vertices $\{u, v, w\}$ and adding the edge (u', v'); that is, $G' = (G - \{u, v, w\}) + (u', v')$. Then $\delta_{n(G)} = 3$, $\delta_{\alpha(G)} \geq 1$, and $\delta_{nt(G)} \leq 0$. Moreover, $\Delta(G') \leq 3$ and G' is obstacle-free.*

Proof. It is clear that $\delta_{n(G)} = 3$. Since the two vertices in u', v' are adjacent in G', any maximum independent set I' of G' contains at most one vertex from $\{u', v'\}$. If $u' \notin I'$, then $I' \cup \{u\}$ is an independent set of G. On the other hand, if $v' \notin I'$, then $I' \cup \{v\}$ is an independent set of G. It follows that $\delta_{\alpha(G)} \geq 1$.

Since u' and v' do not share a neighbor, the edge (u', v') is not a triangle edge in G', and hence u' and v' are nontriangle vertices in G'. Since all the vertices removed from G are triangle vertices, we have $\delta_{nt(G)} \leq 0$. Now to show that G' is obstacle-free, it suffices to show that the addition of (u', v') does not create obstacle graphs. Since both u' and v' are nontriangle vertices in G', the addition of (u', v') cannot create a small obstacle. If the addition of (u', v') creates a medium obstacle, then this edge must be one of the two edges between two nontriangle vertices in the medium obstacle. However, this would imply that there are two triangles in G that are adjacent; this contradicts the hypothesis of the lemma. If the addition of the edge (u', v') creates a large obstacle, then this obstacle must be a separate component in the resulting graph. It is easy to see that if this is the case then no two vertices in $\{u', v', w'\}$ share a common neighbor in G (since w' has degree 2 in G', and hence does not belong to the separate component). Therefore, Algorithm **Simplify-I**

INPUT: *A graph G with $\Delta(G) \leq 3$ such that G is obstacle-free*
OUTPUT: *A graph G'*

1. **Repeat until** none of Operations 4.1– 4.7 applies to G:
   pick the *first* operation in Operation 4.1, ..., Operation 4.7 in this order that applies to G and apply it;
2. **return** the resulting graph;

Figure 2: The algorithm **Simplify-I**.

we can substitute the two vertices u' and v' with u' and w', that is, we modify the operation by setting $G' = (G - \{u, v, w\}) + (u', w')$, and now it is easy to see that the modified operation will not create an obstacle graph. Finally, since one neighbor of each of u' and v' was removed from G, we have $\Delta(G') \leq 3$. $\square$

Operation 4.7 *Suppose that no two triangles in G are adjacent, and let (u, v, w) be a triangle in G such that $d(u) = d(v) = d(w) = 3$. Let u', v', and w' be the neighbors of u, v, w, respectively that are nontriangle vertices. If there are two vertices in $\{u', v', w'\}$, say u' and v', that do not share a common neighbor in G, then set $G' := (G - \{u, v, w\}) + (u', v')$.*

Proposition 4.9 *Let G be a graph with $\Delta(G) \leq 3$ such that G is obstacle-free. Let G_1 be the graph resulting from the application of the algorithm* **Simplify-I** *to G. Then the following are true:*

(i) *$\Delta(G_1) \leq 3$ and G_1 is obstacle-free.*

(ii) *Every triangle vertex in G_1 has degree 3 (in G_1).*

(iii) *$\delta_{nt(G)} \leq 0$, and hence $nt(G_1) \geq nt(G)$.*

(iv) *$\delta_{\alpha(G)} \geq \delta_{n(G)}/3$.*

(v) *No two triangles in G_1 share an edge or are adjacent.*

(vi) *If (u, v, w) is a triangle in G_1 then each of u, v, w has exactly one neighbor u', v', w', respectively, that is a nontriangle vertex. Moreover, vertices u', v', w' are distinct, no two of them are adjacent, and every two of them share a neighbor.*

PROOF.

(i) This follows from the fact that $\Delta(G) \leq 3$ and from Lemma 4.2–Lemma 4.8, which state that the graph resulting from the application of each of Operations 4.1– 4.7 has maximum degree at most 3 and is obstacle-free.

(ii) This follows from the fact that Operations 4.1 is not applicable to G_1.

(iii) This follows from Lemmas 4.2–4.8, which state that for each of Operations 4.1–4.7 we have $\delta_{nt(G)} \leq 0$.

(iv) Each of Operations 4.1–4.7 removes 3ℓ vertices, for some integer $\ell \geq 1$, from the graph and guarantees that the size of the maximum independent set in the graph that the operation is applied to is at least larger by ℓ, which is one third of the number of removed vertices, than the size of the maximum independent set of the graph resulting from the application of the operation. Therefore, if Operations 4.1–4.7 are applied to G to obtain G_1, then $\delta_{n(G)}$ vertices where removed from G, and $\delta_{\alpha(G)} \geq \delta_{n(G)}/3$.

(v) Since G_1 is obstacle-free, G_1 does not contain a small obstacle as a subgraph. Since no triangle vertex at this point is of degree 2, it follows that no two triangles in G share an edge (otherwise G would contain a small obstacle).

Since Operation 4.3 is not applicable to G_1, G_1 does not contain a cycle of triangles. Therefore, to show that no two triangles in G_1 are adjacent, it suffices to show that every maximal path of triangles in G_1 contains exactly one triangle.

Proceed by contradiction. Let $T_1, \ldots, T_\ell$, $\ell > 1$, be a maximal path of triangles, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, and u_i is adjacent to v_{i+1}, for $i = 1, \ldots, \ell - 1$. By part *(ii)* of this proposition, all triangle vertices are of degree 3. Since Operation 4.6 is not applicable to G_1, vertex w_ℓ must share a neighbor x with one of the two vertices $\{w_1, v_1\}$, say w_1, and u_ℓ must share a neighbor y with the other vertex v_1. Since Operation 4.5 is not applicable to G_1, x and y must share a neighbor z. Now since Operation 4.4 is not applicable to G_1, $\ell \leq 2$, and since $\ell > 1$, we have $\ell = 2$. But then the subgraph of G_1 induced by the set of vertices $V(T_1) \cup V(T_2) \cup \{x, y, z\}$ is a medium obstacle in G_1, contradicting the fact that G_1 is obstacle-free (part *(ii)* of this proposition).

Therefore, any maximal path of triangles in G_1 contains exactly one triangle, and hence, G_1 does not contain adjacent triangles.

(vi) Let (u, v, w) be a triangle in G_1. Since every triangle vertex in G_1 is of degree 3 (part *(ii)* of this preposition) and no two triangles in G_1 are adjacent or share an edge (part *(v)* of this proposition), each of u, v, w has exactly one neighbor that is a nontriangle vertex; let these neighbors be u', v', w', respectively, and note that since all these vertices are nontriangle vertices, they must be distinct. Since Operation 4.2 is not applicable to G_1, no two vertices in u', v', w' are adjacent. Since no two triangles in G are adjacent or share an edge, and since Operation 4.7 is not applicable to G_1, every two vertices in u', v', w' share a neighbor.

□

4.2 The second phase

Let G_1 be the graph resulting from G after the first phase, that is, after the application of the algorithm **Simplify-I** to G. In the second phase we apply more operations to simplify G_1 further. Each of the operations introduced in the second phase removes a subgraph H from G_1 satisfying a *local ratio* property. More formally, there exists a subset of vertices $S_H \subseteq V(H)$ that is an independent set, and such that the following are true: (1) $\alpha(G_1) \geq |S_H| + \alpha(G_1 - V(H))$, (2) $|S_H| \geq n(H)/3 + nt(H)/42$, and (3) $nt(G_1) = nt(H) + nt(G_1 - V(H))$. Since each of the operations introduced in this phase removes a set of vertices from G_1, by Observation 4.1, the graph resulting after each operation is obstacle-free and has maximum degree at most 3.

By part *(v)* of Proposition 4.9, no two triangles in G_1 share an edge or are adjacent; therefore, any two triangles in G_1 are disjoint. Moreover, by part *(vi)* of Proposition 4.9, every triangle vertex in G_1 is of degree 3 and has exactly one neighbor that is a nontriangle vertex. For a triangle vertex u, we denote its nontriangle neighbor by u'. Note that for two distinct triangle vertices u, v that are not vertices of the same triangle, u' can be identical to v'. For any triangle (u, v, w) in G_1, by part (vi) of Proposition 4.9, the vertices u', v', w' are distinct, no two of them are adjacent, and every two of them share a common neighbor that is a nontriangle vertex. Note that the three vertices u', v', w' could share the same common neighbor.

Let u' be a vertex that is adjacent to some triangle vertex. We claim that u' has exactly one neighbor that is a triangle vertex, unless the graph G_1 has a component of exactly 10 vertices and an independent set of size 4. In effect, let (u, v, w) and (p, q, r) be two distinct triangles such that u' is a neighbor of both u and p. Then u' must share a common neighbor with each of v', w', q', r'. Since u, v, w are distinct vertices, and p, q, r are distinct vertices, and since the degree of u' is at most 3, this is only possible if $q' = v'$ and $r' = w'$ (or $q' = w'$ and $r' = v'$) and there exists a nontriangle vertex x that is adjacent to u', v', w'. In this case the degree of each of the vertices $u, v, w, p, q, r, u', v', w', x$ in G_1 must be 3, and hence the subgraph H of G_1 induced by these vertices must be a connected component of G_1. It is easy to see that the set of vertices $\{p, u, v', w'\}$ is an independent set in H of size 4. Since $n(H) = 10$, $nt(H) = 4$, and $\alpha(H) = 4$, it follows in this case that $\alpha(H) \geq n(H)/3 + nt(H)/42$. We call such components in G_1 *special components*; see Figure 6 in Section 7 for illustration. Based on the above discussion, we introduce the following operation:

Operation 4.8 *For every special component H in G_1, set $G_1 := G_1 - V(H)$.*

We assume in the rest of this subsection that Operation 4.8 is not applicable to G_1.

From the above discussion, every triangle in G_1 must be contained in one of the two subgraphs depicted in Figure 7 in Section 7; we call the graph on the left a *type-I steeple* and the one on the right a *type-II steeple*. We call the vertices u, v, w the *triangle* vertices of the steeple, and vertices u', v', w' the *middle* vertices of the steeple; we call vertex x in a type-I steeple and vertices x, y, z in a type-II steeple the *top* vertices of the steeple. Note that no edge exists between two middle vertices of a steeple. Moreover, since no two triangles in G_1 are adjacent, no edge exists between two top vertices of a type-II steeple. Therefore, G_1 contains a steeple as a subgraph if and only if it contains it as an induced subgraph. The vertices u', v', w' in a type-I steeple can be of degree 2 or 3 in the graph, and so can the vertices x, y, z in a type-II steeple. Moreover, the vertices u', v', w' in a type-I steeple can have a common neighbor(s) other than x.

Since G_1 does not contain special components, and since the top vertex of a type-I steeple is of degree 3, it is easy to see that if two type-I steeples in G_1 are not vertex-disjoint, then they must share the same triangle and middle vertices, that is, the three middle vertices of a steeple must share another common neighbor besides the top vertex of the steeple; such steeples will be dealt with later by Operation 10. Now any type-I steeple must be vertex-disjoint from any type-II steeple. This can be seen as follows. Let S_1 be a type-I steeple and S_2 be a type-II steeple, and suppose, to get a contradiction, that $V(S_1) \cap V(S_2) \neq \emptyset$. If S_1 and S_2 have a triangle vertex in common, then they must share the same triangle, and consequently, the same middle vertices. It is easy to see now that there must exist two top vertices in S_2 that are distinct from the top vertex of S_1, and such that each of them is adjacent to two middle vertices of S_1; this is impossible. Assume now that the triangle in S_1 is distinct from that in S_2. Therefore, the middle vertices of S_1 are distinct from the middle vertices of S_2. Clearly, the top vertex of S_1 has to be distinct from any middle or top vertex in S_2, and consequently, S_1 and S_2 are disjoint. We conclude that every type-I steeple is disjoint from every type-II steeple in G_1. Finally, if two distinct type-II steeples in G_1 overlap, then by the same token as above, they cannot overlap on triangle vertices, and their triangles must be disjoint. Consequently, no middle vertex in one of the steeples can be a middle vertex in the other steeple. However, it is possible that a middle vertex in one of the steeples is a top vertex in the other. If this is the case, then all the middle vertices of one steeple must be the top vertices of the other steeple, and G_1 contains a component that is a large obstacle (see Figure 1). This contradicts the fact that G_1 is obstacle-free. We conclude that any two steeples in G_1 are vertex-disjoint unless they are two type-I steeples that share the same triangle and middle vertices, and their top middle vertices all share two common neighbors.

We now apply more operations to simplify G_1. Those operations are depicted in Figure 8. Each of these operations removes a subgraph H from G_1, which is the subgraph induced by the set of solid/black vertices plus their neighbors (the set of gray vertices in the figure), to obtain a subgraph $G_1 - V(H)$ of G_1 such that there exists a subset of vertices $S_H \subseteq V(H)$ that is an independent set (the set of black vertices), and such that the following holds true: (1) $\alpha(G_1) \geq |S_H| + \alpha(G_1 - V(H))$, (2) $|S_H| \geq n(H)/3 + nt(H)/42$, and (3) $nt(G_1) = nt(H) + nt(G_1 - V(H))$. Before we describe briefly the role of each of these operations, we give the following definition.

Definition 4.1 Let S and S' be two steeples. The *distance* between S and S' is defined as follows. If S and S' are not vertex disjoint, then the distance between S and S' is zero. Otherwise, the distance between S and S' is the length of a shortest path between a middle vertex of S and a middle vertex of S' if both S and S' are type-I steeples, the length of a shortest path between a middle vertex of S and a top vertex of S' if S is a type-I steeple and S' is a type-II steeple, or the length of a shortest path between a top vertex of S and a top vertex of S' if both S and S' are type-II steeples.

The graph K_4^* is K_4 with two of its edges each subdivided twice (see the third graph from the left in Figure 9).

Operation 10 removes any type-I steeple in which two middle vertices share a neighbor other than the top vertex of the steeple; consequently, this operation removes any two type-I steeples that share the same triangle and middle vertices, but have different top vertices. Operation 11 removes any type-I steeple in which a middle vertex has a degree-2 neighbor. Operation 12 removes any type-I steeple such that there exists an edge between the neighbors of two of its middle vertices. Operation 13 removes any type-II steeple such that two of its top vertices share a neighbor other than a middle vertex in the steeple. Operations 14, 15, and 16 remove any two type-I steeples of distance 1 (i.e., at most one edge apart). Operation 17 removes any two type-I steeples of distance 2, and Operation 18 removes any two type-I steeples of distance 3. Operation 19 removes any two type-II steeples of distance 1. Operation 20 removes any two type-II steeples of distance 2. Operations 21, 22, and 23 remove any type-I and type-II steeples whose distance is 1. Operation 24 removes any type-I and type-II steeples whose distance is 2. Operation 25 removes any type-I and type-II steeples whose distance is 3. Operation 26 removes any type-II steeple such that a top vertex in the steeple has a neighbor of degree 2. Operation 27 removes any C_5 having two degree-2 nonadjacent vertices. Operation 28 removes any K_4^* with a degree-2 vertex. Operation 29 removes any type-II steeple and any C_5 such that there are two edges between two top vertices of the steeple and two nonadjacent vertices of the C_5. Finally, Operation 30 removes any type-II steeple and any K_4^* such that there is an edge between a top vertex of the steeple and a vertex of the K_4^*.

It is possible that after the application of the operations in Operation 10–30, that some operation from the first phase becomes applicable. However, only two operations from the first phase may become applicable: Operation 4.1 and Operation 4.7. If this is the case, we apply these operations again. To formalize the above discussion, let G_1 be the resulting graph from G after the first phase. In the second phase we apply the following algorithm **Simplify-II** to G_1.

Algorithm **Simplify-II**

INPUT: *A graph* G_1 *with* $\Delta(G_1) \leq 3$ *such that* G_1 *is obstacle-free and* **Simplify-I** *is not applicable to* G_1
OUTPUT: *A graph* G_2

1. **While** Operation 4.8 applies to G_1 apply it;
2. **Repeat until** none of Operations 4.1, Operation 4.7, or Operations 10–30 applies to G_1:
   pick the *first* operation in Operation 4.1, Operation 4.7, Operation 10, Operation 11, ..., Operation 30 in this order that applies to G_1 and apply it;
3. **return** the resulting graph;

Figure 3: The algorithm **Simplify-II**.

Proposition 4.10 *Let* G_2 *be the graph resulting from the application of the algorithm* **Simplify-II** *to* G_1*. The following are true:*

(i) Every triangle in G_2 *appears either in a type-I or a type-II steeple.*

(ii) The distance between any type-I steeple and any other steeple in G_2 *is at least 4, and the distance between any two type-II steeples in* G_2 *is at least 3.*

(iii) For any type-I steeple in G_2*, the neighbors of its middle vertices are all distinct degree-3 vertices, and no two of them are adjacent.*

(iv) For any type-II steeple in G_2*, the neighbors of its top vertices are all distinct degree-3 vertices.*

(v) For any type-II steeple in G_2*, none of its top vertices is adjacent to a vertex of a* K_4^**, and no two of its top vertices are adjacent to two nonadjacent vertices of a* C_5*.*

(vi) No C_5 *in* G_2 *has two degree-2 nonadjacent vertices, and no* K_4^* *in* G_2 *has a degree-2 vertex.*

(vii) If $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$ *then* $\alpha(G) \geq n(G)/3 + nt(G)/42$*.*

PROOF.

(i) Since Operations 4.1– 4.7 do not apply to G_2 (Operations 4.1 and Operations 4.7 do not apply to G_2 by the algorithm **Simplify-II**, and Operations 4.2– 4.6 do not apply to G_2 because they do not apply to G_1), it follows from part (*vi*) of Proposition 4.9 and the discussion at the beginning of this subsection that every triangle in G_2 is contained in either a type-I or a type-II steeple.

(ii) The fact that the distance between any type-I steeple and any other steeple is at least 4 follows from the fact that G_2 does not contain special components, and the fact that none of Operation 10, Operations 14–18, and Operations 21–25 is applicable to G_2; the fact that the distance between any two type-II steeples is at least 3 follows from the fact that G_2 is obstacle-free and that none of Operations 19–20 is applicable to G_2.

(iii) This follows from the fact that none of Operations 10–12 is applicable to G_2.

(iv) This follows from the fact that none of Operations 13 or 26 is applicable to G_2.

(v) This follows from the fact that none of Operations 28 or 30 is applicable to G_2.

(vi) This follows from the fact that none of Operations 27 or 29 is applicable to G_2.

(vii) From part (vi) of Proposition 4.9 we have:

$$\alpha(G) \quad \geq \quad \alpha(G_1) + (n(G) - n(G_1))/3. \tag{1}$$

Now each of the operations applied in the second phase, including Operation 4.1 and Operation 4.7, removes a set of vertices $V(H)$ from G_1 to obtain a graph $G_1 - V(H)$ such that $\alpha(G_1) \geq \alpha(G_1 - V(H)) + |S|/3 + nt(S)/42$. Moreover, none of these operations creates new triangle vertices. (Note that Operation 4.7 adds a new edge but does not create new triangle vertices, as proved earlier.) Therefore, by additivity, at the end of the second phase we have:

$$\alpha(G_1) \quad \geq \quad \alpha(G_2) + (n(G_1) - n(G_2))/3 + (nt(G_1) - nt(G_2))/42. \tag{2}$$

By part (iii) of Proposition 4.9, we have $nt(G_1) \geq nt(G)$. Combining the last inequality with Inequalities (1) and (2) we obtain:

$$\alpha(G) \quad \geq \quad \alpha(G_2) + (n(G) - n(G_2))/3 + (nt(G) - nt(G_2))/42. \tag{3}$$

Now if $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$, then from Inequality (3) it follows that $\alpha(G) \geq n(G)/3 + nt(G)/42$.

□

4.3 The third phase

Let G_2 be the resulting graph after the second phase. Then G_2 satisfies all the properties described in Proposition 4.10. By part (vii) of Proposition 4.10, it suffices to show that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$ to conclude that $\alpha(G) \geq n(G)/3 + nt(G)/42$. In this subsection we proceed to do just that. We summarize below the utilized approach.

As in the previous two phases, we will apply some operations to G_2. The purpose of the operations applied in this phase is the removal of all triangles from G_2. Each of these operations removes a subgraph H of G_2; however, in contrast to the operations performed in phase 2, the removed subgraph does not satisfy the "local ratio" property, namely that we can always add to any independent set of $G - V(H)$ an independent set of H of cardinality at least $n(H)/3 + nt(H)/42$. To show that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$, we use a charging argument to measure the impact of each of these operations on the resulting graph, in addition to amortized analysis. We first introduce the necessary notations.

A block of a graph is called *difficult* [10] if it is isomorphic to one of the following four graphs (see Figure 9 in Section 9 for illustration): K_3, C_5, K_4 with two of its edges each subdivided twice, denoted K_4^*, and a graph arising from C_5 by adding a new vertex and connecting it to three consecutive vertices of C_5. A connected graph is called *bad* [10] if every block of the graph is either a difficult block or an edge between two difficult blocks.

Harant et al. [10] showed that if a graph G' is a K_4-free graph with $\Delta(G') \leq 3$ then $\alpha(G') \geq (4n(G') - e(G') - \lambda(G') - tr(G'))/7$, where $\lambda(G')$ is the number of components of G' that are bad, and $tr(G')$ is the number of vertex-disjoint triangles in G'. It follows that:

Lemma 4.11 *Let G' be a triangle-free graph such that $\Delta(G') \leq 3$. If G' does not contain bad components then $\alpha(G') \geq (4n(G') - e(G'))/7$.*

Definition 4.2 Let H be a subgraph of G_2. Call an edge e with exactly one endpoint in H a *fringe edge* to H. Call a vertex $v \in V(H)$ a *boundary vertex* if v is an endpoint of a fringe edge to H; otherwise, call v an *internal vertex* of H. Let $e^+(H)$ denote the number edges in H plus the number of fringe edges to H (that is, the number of edges with at least one endpoint in H).

Suppose that we can apply some operations to G_2 to remove all triangles from G_2 such that the following conditions are satisfied: (1) each operation removes a subgraph H such that there exists an independent set consisting of internal vertices in H of size at least $(23n(H) - 6e^+(H) + nt(H))/42$; and (2) the subgraph resulting from G_2 at the end of these operations is triangle-free and contains no bad components. Suppose that these operations remove a subgraph G_2^- from G_2. Then by Lemma 4.11, we have $\alpha(G_2 - V(G_2^-)) \geq (4n(G_2 - V(G_2^-)) - e(G_2 - V(G_2^-))/7 = (23n(G_2 - V(G_2^-)) - 6e^+(G_2 - V(G_2^-)) + nt(G_2 - V(G_2^-))/42$; the last equality is true because $G_2 - V(G_2^-)$ is triangle-free, and hence $nt(G_2 - V(G_2^-)) = n(G_2 - V(G_2^-))$. Since the operations performed satisfy condition (1) above, we can add to any independent set of $G_2 - V(G_2^-)$ an independent set of G_2^- of size at least $(23n(G_2^-) - 6e(G_2^-) + nt(G_2^-))/42$. We conclude that the independence number of G_2 satisfies: $\alpha(G_2) \geq (23n(G_2) - 6e^+(G_2) + nt(G_2))/42$. Since $\Delta(G_2) \leq 3$, $e(G_2) \leq 3n(G_2)/2$, and consequently, $\alpha(G_2) \geq (14n(G_2) + nt(G_2))/42 = n(G_2)/3 + nt(G_2)/42$.

It follows from the above discussion that it is sufficient to show that each of the operations that we apply satisfies conditions (1) and (2) above. For a subgraph H of G_2, let $\phi(H) = |S_H| - (23n(H) - 6e^+(H) + nt(H))/42$, where S_H is a maximum independent set consisting of internal vertices in H. Then an operation that removes a subgraph H such that $\phi(H) \geq 0$ satisfies condition (1) above. We would like to show that each introduced operation that removes a subgraph H satisfies $\phi(H) \geq 0$. This will be the case for most of the operations that we apply except few. To circumvent this issue, we use amortized analysis: we show that each time one of these few operations applies, the "deficit" in the function ϕ caused by this operation can be "compensated for" by operations that *must* have occurred earlier in this phase. To implement this concept, for each operation that removes a subgraph H, we introduce a parameter $c(H)$, where $c(H)$ is the *cost* (or debit) of operation H meant to possibly pay off the deficit of some later operations. We have the following definition:

Definition 4.3 *Let H be a subgraph of G_2, and let S_H be a maximum independent set in H consisting of internal vertices to H. Let $E_1(H)$ be the set of fringe edges to H whose endpoint in $G_2 - V(H)$ is a neighbor of a top vertex in some type-II steeple, and let $E_2(H)$ be the set of remaining fringe edges to H. Let $s = 1/14$. Define the functions $\phi(H) = |S_H| - (23n(H) - 6e^+(H) + nt(H))/42$ and $\Phi(H) = \phi(H) - c(H)$, where $c(H) = (s/2)|E_1(H)| + (s/4)|E_2(H)|$. We define the auxiliary function $\phi^-(H) = \alpha_H - (23n(H) - 6e(H) + nt(H))/42$. ($\phi^-(H)$ is a variant of $\phi(H)$ that is "local" to H, where S_H is replaced with a maximum independent set of H and the fringe edges to H are excluded.)*

Our task now becomes to remove all triangles in G_2 by applying operations, each of which removes a subgraph H from the graph, such that the sum of $\Phi(H)$ over all operations is nonnegative. At each point we consider a triangle in the resulting graph. This triangle will always be contained in a type-I or a type-II steeple. At the beginning, the previous statement is true because G_2 satisfies part (i) of Proposition 4.10. The statement remains true since by part (ii) of Proposition 4.10 any two steeples are of distance at least 3. Since every triangle in G_2 is contained in a steeple, and since each steeple is 2-connected, no bad component of G_2 contains a triangle (refer to Figure 9), and every bad component C in G_2 is triangle-free. Now if a component C of G_2 is triangle-free, then by [11], we have $\alpha(C) \geq 5n(C)/14 = n(C)/3 + n(C)/42$. Therefore, since our goal is to prove that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$, by additivity, we can assume that at the beginning of this phase every component of G_2 contains some triangle, and that G_2 does not contain bad components. We will ensure that none of the operations applied to G_2 in this phase introduces bad components. To do so, whenever an operation introduces bad components we will remove them and consider the original operation, plus the removal of the bad components, as a single operation. Since by part (ii) of Proposition 4.10 any two steeples are of distance at least 3, and since each of the introduced operations only affects the close vicinity of a steeple, it will follow that after each operation every triangle in the resulting graph is still contained in a steeple. This implies that any bad component resulting from an operation is triangle-free, and hence, each of its blocks must be either a C_5 or a K_4^* block, or an edge between two such blocks.

Our goal is to show, for each of the proposed operations that removes a subgraph H from G_2, that $\Phi(H) \geq 0$. Since the removal of H may create bad components, the removal of these bad components needs to be accounted for. The following lemmas will be useful for that purpose.

Lemma 4.12 *Let H' be a subgraph of a graph G'. Let $H_0, \ldots, H_t$, $t \geq 0$, be subgraphs of H' such that: (1) $V(H_0), \ldots, V(H_t)$ is a partition of $V(H')$; (2) a vertex in $V(H_i)$, $i = 0, \ldots, t$, is a nontriangle vertex in G' if and only if it is a nontriangle vertex in H_i; (3) every fringe edge to H' is a fringe edge to H_0; (4) no edge exists between a vertex in $V(H_i)$ and a vertex in $V(H_j)$ for $0 < i < j$; and (5) there exists an independent set I_0 of internal vertices of H_0, and maximum independent sets I_i of H_i, $i = 0, \ldots, t$, such that $I_0 \cup I_1 \cup \ldots \cup I_t$ is an independent set of H'. Then $\phi(H') \geq \phi(H_0) + \Sigma_{i=1}^{t} \phi^-(H_i)$. Moreover, equality holds if there exists a maximum independent set I of internal vertices of H' such that $I \cap V(H_0)$ is an independent set of internal vertices of H_0.*

Lemma 4.13 *Let B be a graph that is a bad component and in which each block is either a C_5 or a K_4^*, or an edge between two such blocks. Then $\phi(B) = \phi^-(B) = -1/7$.*

Proof. Since no edge is a fringe edge to B we have $e^+(B) = e(B)$. Moreover, a maximum independent set of internal vertices of B is simply a maximum independent set of B. It follows from the previous statements that $\phi(B) = \phi^-(B)$. Therefore, it suffices to show that $\phi(B) = -1/7$.

The proof is by induction on the number of blocks in B. (Note that B is a tree of blocks.) The base case is when B consists of a single leaf-block, and in this case it can be easily verified that the statement is true in each of the cases when B is a C_5 and when B is a K_4^*.

Suppose now that B consists of more than one block, and suppose inductively that the statement is true for any bad component with fewer blocks than B. Let L be a leaf-block in B. Then there exists exactly one edge between L and $B - L$. Since any maximum independent set of B contains exactly 2 vertices of L if $L = C_5$, and 3 vertices of L if $L = K_4^*$, which is the size of a maximum independent set of L consisting of internal vertices, by Lemma 4.12, we have $\phi(B) = \phi(L) + \phi^-(B - V(L))$. Since $B - V(L)$ is a graph that is a bad component with fewer blocks than B, by the inductive hypothesis we have $\phi(B - V(L)) = \phi^-(B - V(L)) = -1/7$. Since L has one fringe edge (to $B - V(L)$), it can be easily verified that $\phi(L) = 0$, in both cases when L is a C_5 and when L is a K_4^*. It follows that $\phi(B) = -1/7$. □

Corollary 4.14 *Let H' be a subgraph of a graph G', and suppose that B is a bad component in $G' - V(H')$. Then $\phi^-(B) = -1/7$.*

PROOF. This follows from Lemma 4.13 since the function ϕ^- is local to B, and hence its value is the same as when B is the whole graph. $\square$

Lemma 4.15 *Suppose that G_2 does not contain bad components and let H be a subgraph of G_2 such that $G_2 - V(H)$ contains bad components. Let B be a bad component in $G_2 - V(H)$.*

(i) Suppose that B consists of a single block L. If $L = C_5$ and there are at most two (fringe) edges between the vertices of L and the (boundary) vertices of H, or if $L = K_4^*$ and there are at most three edges between the vertices of L and the vertices of H, then we can apply an operation that removes B to obtain $G_2 - V(B)$ such that $\Phi(B) \geq 0$.

(ii) If B contains a leaf-block L that is a C_5 such that there is at most 1 edge between the vertices of L and the vertices of H then we can apply an operation to L that removes it to obtain $G_2 - V(L)$ such that $\Phi(L) \geq 0$; and if B contains a leaf block L that is a K_4^* such that there are at most 2 edges between the vertices of L and the vertices of H then we can apply an operation to L that removes it to obtain $G_2 - V(L)$ such that $\Phi(L) \geq 0$.

PROOF. It is easy to verify that in each of the cases described above we can find an independent set of L, consisting of internal vertices, of size 2 when L is a C_5 and of size 3 when L is a K_4^*.

Since L is either a C_5 or a K_4^*, by Corollary 4.14 we have $\phi^-(L) = -1/7$. If there are at least two fringe edges e, e' to L, then since each edge contributes $1/7 = 2s$ to $\phi(L)$ and has cost of at most $s/2 = 1/28$, it is easy to verify that the operation that removes L satisfies $\Phi(L) \geq 0$.

Suppose now that there is exactly one fringe edge to L. The operation that removes L satisfies $\phi(L) = 0$. However, since there is a fringe edge to L, we incur a cost $c(L) \geq -s/2$, and $\Phi(L) \geq -s/2$. Now at the beginning of the third phase, by parts (v) and (vi) of Proposition 4.10, L must have had at least three fringe edges if L is a C_5, and 4 fringe edges if L is a K_4^*. In either case, at least 2 fringe edges to L were removed since the start of the third phase, and this must have been due to the application of some earlier operations in the third phase. Since the removal of each such edge has been paid for at a cost of at least $s/4$ by a previous operation, we can associate with L a surplus of at least $s/2$. This surplus can now be used to pay for the cost of removing the fringe edge to L. Therefore, we conclude that the operation of removing L satisfies $\Phi(L) \geq 0$. $\square$

Lemma 4.15 suggests the following procedure. Suppose that G_2 does not contain bad components and let H be a subgraph of G_2 that contains a steeple. If a leaf-block L in a bad component B in $G_2 - V(H)$ satisfies one of the statements in Lemma 4.15, then let L be such a block with the fewest number of fringe edges between its vertices and H among all leaf-blocks of B. Then L can be removed to obtain $G_2 - V(L)$ such that $\Phi(L) \geq 0$; moreover, by the choice of L and the fact that H contains a steeple, $G_2 - V(L)$ does not contain bad components. Hence, by the additivity of the function Φ, L can be removed from the bad component and the operation that removes L accounts for itself; in addition, the resulting graph does not contain bad components. We say in this case that we "peel" L. We introduce a subroutine **Peel** that repeatedly removes a leaf-block satisfying the statement of Lemma 4.15 until no leaf-block in a bad component of $G_2 - V(H)$ satisfies the statement of the lemma. In such case the following holds true:

Lemma 4.16 *Suppose that G_2 does not contain bad components and let H be a subgraph of G_2 that contains a steeple. Suppose further that* **Peel** *does not apply. Let B be a bad component in $G_2 - V(H)$. If B consists of a single block that is a C_5, then there are at least three edges between* *the vertices of B and the (boundary) vertices of H, and if B consists of a single block that is a K_4^* then there are 4 edges between the vertices of B and those in H. If B is not a leaf-block, then for each leaf-block L in B that is a C_5, there are at least two edges between the vertices of L and those of H, and for each leaf-block L in B that is a K_4^* there are at least three edges between its vertices and those of H.*

We now introduce the operations to be applied to G_2 to remove all triangles. We group these operations into two categories depending on whether the triangle to which the operation is applied is contained in a type-I or a type-II steeple. From the above discussion, we can assume that for every subgraph H in the resulting graph, **Peel** does not apply to any leaf-block in a bad component of $G_2 - V(H)$. Note that since **Peel** may remove a subgraph from the graph, **Peel** may affect the degrees of some of the vertices in H, and hence, in the steeple that will be contained in H. As a matter of fact, **Peel** can only affect the degrees of the middle vertices in a type-I steeple and the top vertices in a type-II steeple. However, since each of these vertices has degree 2 within the steeple it is contained in, the degree of any such vertex after **Peel** is applied will be at least 2. Moreover, since each of the operations removes a subgraph, the properties about the vertices of the steeples, described in Proposition 4.10, except for those pertaining to the degrees, remain true in the resulting graph after each operation; in particular, the middle vertices in a steeple form an independent set, the top vertices of a type-II steeple form an independent set, the neighbors of the middle vertices in a type-I steeple are distinct (part (v) of the proposition), etc.

The basic operation that we would like apply to a steeple can be described as follows. We consider an independent set S_H in the steeple, and form the subgraph H whose vertices consist of the vertices of the independent set plus their neighbors. The steeple will always be contained in H, but the chosen independent set will depend on the structure of the steeple. We will then remove H to obtain $G_2 - V(H)$ and verify that $\Phi(H) \geq 0$. In most cases we will end up upper bounding the value of $c(H)$ in $\Phi(H)$.

Operations on type-I steeples

Let (u, v, w) be a triangle that is contained in a type-I steeple S; let u', v', w' be the middle vertices of the steeple, and let x the top vertex of the steeple, as illustrated in Figure 7 in Section 7. We would like to apply an operation that removes triangle (u, v, w). We will distinguish several cases based on the structure of S. In each of these cases, the operation will remove a subgraph H containing S (and hence u, v, w); however, the operation will not affect any other steeples in the graph, as can be seen from the operations below. That is, any remaining triangle in the resulting graph will still be contained within a steeple. This is true because any two steeples in G_2 are of distance at least 3 from each other (Proposition 4.10), and this remains true after each operation since each operation results in a subgraph of the original graph.

The following cases exhaustively consider all possible structures of S. We note that some of these operations may not originally apply to G_2, but may apply later after some steeples have been removed. Also, the operations are considered in the listed order; that is, when applying an operation, we assume that none of the previous operations applies. Without loss of generality, we will call the resulting graph after each operation G_2. We refer the reader to Figure 13 in Section 7 for illustration. In each figure in Figure 13 in Section 7, the set of black vertices constitutes S_H, and the set of black and grey vertices constitutes H.

Case 4.17 *S has at least two middle vertices of degree 2.*

Please refer to the figure Case 4.17 in Figure 10 in Section 7 for illustration. Suppose that u', v' are of degree 2. Let $S_H = \{u', v', w\}$, and let H be the subgraph induced by the set of vertices in S_H and their neighbors; that is, $H = S$ in this case. Since there is at most one fringe edge to H (incident to w'), and since **Peel** does not apply, by Lemma 4.16, there is no bad component in $G_2 - V(H)$. Since there can be at most one fringe edge (w', p) to H, where $p \notin H$, and since the distance between a type-I steeple and any other steeple is at least 4 by part (ii) of Proposition 4.10, p cannot be a neighbor of a top vertex in a type-II steeple, and hence the cost of edge (w', p) (if it exists), which is equal to $c(H)$ since this edge is the only possible fringe edge to H, is $s/4$. Therefore, we have $n(H) \leq 7$, $nt(H) \leq 4$, $e^+(H) \geq 9$, and $c(H) \leq s/4$. Plugging all these values in $\Phi(H)$ we get $\Phi(H) \geq 0$.

Case 4.18 *S has exactly one middle vertex of degree 2.*

Suppose that u' is of degree 2. Let $S_H = \{u', v', w\}$, and let H be the subgraph induced by the set of vertices in S_H and their neighbors. Since there are at most 3 fringe edges to H, and since **Peel** does not apply, by Lemma 4.16 there can be at most one leaf-block in $G_2 - V(H)$, and hence there can be at most one bad component B that is a leaf-block. Moreover, this leaf-block must be a C_5, and all 3 fringe edges to H must be incident on this leaf-block. Therefore, $G_2[V(H) \cup V(B)]$ forms a separate component C of G_2. Let I be an independent set of 5 vertices consisting of 2 vertices of B plus S_H. Clearly $c(C) = 0$. In this case we have $n(C) \leq 13, nt(C) \leq 10$, and $e^+(C) \geq 18$. It follows that $\Phi(C) \geq 0$. Please refer to the figure Case 4.18–(A) in Figure 10 in Section 7.

Suppose now that $G_2 - V(H)$ contains no bad components; this case is depicted in the figure Case 4.18–(B) in Figure 10 in Section 7. We have $n(H) \leq 8$, $nt(H) \leq 5$, and $e^+(H) \geq 11$. Note that since the distance between a type-I steeple and any other steeple is at least 4, none of the fringe edges to H could be incident on a neighbor of a top vertex in a type-II steeple, and hence, $c(H) \leq 3s/4$. This case results in $\Phi(H) \geq 0$.

Case 4.19 *At least one neighbor of a middle vertex in S is of degree 1.*

Suppose that u' has a degree-1 neighbor u''. Let $S_H = \{u'', v', w', u\}$, and let H be the subgraph of G_2 induced by S_H and its neighbors; note that there are at most 4 fringe edges to H incident on the neighbors of v' and w'. Since **Peel** does not apply, if there are bad components in $G_2 - V(H)$, then by Lemma 4.16 there can be at most one bad component B, and at least 3 fringe edges to H are incident on vertices in B. Since S_H is independent from any maximum independent set of B, by Lemma 4.12 (applied with $H_0 = H$ and $H_1 = B$), we have $\phi(G_2[V(H) \cup V(B)]) \geq \phi(H) + \phi^-(B)$. By Corollary 4.14, $\phi^-(B) = -1/7$. Since at least 3 edges are fringe edges to H, we have $e^+(H) \geq 15$. Moreover, since at most one fringe edge to H can be incident on a vertex in $(G_2 - V(H)) - V(B)$, $c(H) \leq s/4$. Therefore, we have $n(H) \leq 10$, $nt(H) \leq 7$, $e^+(H) \geq 15$, and $c(H) \leq s/4$. It follows in this case that $\phi(H) > 1/7 + s/4$, and $\Phi(G_2[V(H) \cup V(B)]) \geq 0$. Please refer to the figure Case 4.19–(A) in Figure 10 in Section 7.

If $G_2 - V(H)$ contains no bad components, then since none of the previous cases applies, every middle vertex of S has degree 3, and $e^+(H) \geq 12$. Moreover, at most 4 edges are fringe to H, and hence $c(H) \leq s$. It follows in this case that $\Phi(H) \geq 0$. Please refer to the figure Case 4.19–(B) in Figure 10 in Section 7.

Case 4.20 *At least two neighbors of the middle vertices in S are of degree 3.*

Let $S_H = \{u', v', w\}$, and assume that the non-steeple neighbors u'', v'' of u', v', respectively, have degree 3. Let H be the subgraph of G_2 induced by S_H and its neighbors. There are at most

5 fringe edges to H: 4 edges incident on u'', v'', and an edge incident on w'. Since **Peel** does not apply to $G_2 - V(H)$, by Lemma 4.16, there can be at most one bad component in $G_2 - V(H)$.

If there is a bad component B in $G_2 - V(H)$, by Corollary 4.14, $\phi^-(B) = -1/7$. Since S_H is independent from any maximum independent set of B, we have $\phi(G_2[V(H) \cup V(B)]) \geq \phi(H) + \phi^-(B)$. Since w', u'', v'' are all of degree 3, and since by part (iii) of Proposition 4.10 the neighbors of the middle vertices of a type-I steeple are distinct and no two of them are adjacent, we have $e^+(H) \geq 16$. Moreover, since at least 3 fringe edges to H are incident on vertices in B, at most 2 edges exist between H and $(G_2 - V(H)) - V(B)$, and none of those edges can be incident on a neighbor of a top vertex in a type-II steeple; therefore, $c(H) \leq s/2$. We have $n(H) \leq 9$, $nt(H) \leq 6$, $e^+(H) \geq 16$, and $c(H) \leq s/2$. It follows in this case that $\phi(H) > 1/7 + s/2$, and $\Phi(G_2[V(H) \cup V(B)]) \geq 0$. Please refer to the figure Case 4.20–(A) in Figure 10 in Section 7.

If $G_2 - V(H)$ contains no bad components, then since at most 5 edges are fringe edges to H, $c(H) \leq 5s/4$. We have $n(H) \leq 9$, $nt(H) \leq 6$, $e^+(H) \geq 16$, and $c(H) \leq 5s/4$. It follows in this case that $\Phi(H) \geq 0$. Please refer to the figure Case 4.20–(B) in Figure 10 in Section 7.

Case 4.21 *Exactly one neighbor of the middle vertices in S is of degree 3.*

Let $S_H = \{w', v', u\}$, and assume that the non-steeple neighbor w'' of w' has degree 3. Let u'', v'' be the non-steeple neighbors of u' and v' and note that u'' and v'' are of degree 2. Let H be the subgraph induced by S_H and its neighbors. Note that by part (iii) of Proposition 4.10, no two vertices in u'', v'', w'' are adjacent. There are 4 fringe edges to H: 2 edges incident on w'', an edge incident on v'', and an edge incident on u' (note that at this point all neighbors of the middle vertices of S have degree at least 2). Since **Peel** does not apply to $G_2 - V(H)$, by Lemma 4.16, there can be at most one bad component in $G_2 - V(H)$.

Suppose that there is a bad component B in $G_2 - V(H)$. If B consists of a single block that is a K_4^*, then all fringe edges to H must be incident on the vertices in B, and $G_2[V(H) \cup V(B)]$ must form a separate component C of G_2. Then we can choose an independent set of size 7 of the whole component C, as illustrated in the figure Case 4.21–(A) in Figure 11 in Section 7, and remove the whole component. Note that since C is a component, $c(C) = 0$. We have $n(C) \leq 18$, $nt(C) = 15$, $e^+(C) \geq 26$. In this case we have $\Phi(C) \geq 0$.

If B consists of a single block that is a C_5, then at least 3 fringe edges to H must be incident on B. Therefore, either v'' or u'' must be adjacent to a vertex in B; suppose without loss of generality that v'' is. Then we can always pick an independent set I of size 6 consisting of two vertices of B plus $\{v'', w', u', v\}$. Let H_I be the subgraph induced by the vertices in I and their neighbors. Since at least 3 fringe edges to H are incident on vertices in B, there can be at most 1 fringe edge to H_I (incident on u'' or w''). Since **Peel** does not apply to any subgraph of G_2, there can be no bad components in $G_2 - V(H_I)$. We have $n(H_I) \leq 15$, $nt(H_I) \leq 12$, $e^+(H_I) \geq 21$, and $c(H_I) \leq s/4$. It follows in this case that $\Phi(H_I) \geq 0$. Please refer to the figure Case 4.21–(B) in Figure 11 in Section 7.

If B does not consist of a single block, then B must contain exactly two leaf-blocks, and hence B must be a chain of blocks; please refer to the figure Case 4.21–(C) in Figure 11 in Section 7. Moreover, since there are 4 fringe edges to H, both leaf-blocks in B must be C_5 blocks, each having two fringe edges to H incident on it. Hence, $G_2[V(H) \cup V(B)]$ must form a separate component C of G_2. By Corollary 4.14, $\phi^-(B) = -1/7$. Observe that we can always choose a maximum independent set of B that avoids the neighbor of a vertex in u'', v'', w'', say u''. Let $I = \{u'', v', w', u\}$ and let H_I be the subgraph of G_2 induced by the vertices in $S \cup \{u'', v'', w''\}$. By Lemma 4.12 (note in this case that I does not consist of internal vertices of H_I, however, this does not matter since the maximum independent set of B avoids the neighbor of u'') we have $\phi(G_2[V(H_I) \cup V(B)]) \geq \phi(H_I) + \phi^-(B)$. We have $n(H_I) \leq 10, nt(H_I) \leq 7, e^+(H_I) \geq 16$, and $c(H_I) = 0$. It follows that $\phi(H_I) > 1/7$, and $\Phi(C) \geq 0$.

If $G_2 - V(H)$ contains no bad components, then since at most 4 edges are fringe edges to H, $c(H) \leq s$. We have $n(H) \leq 9$, $nt(H) \leq 6$, $e^+(H) \geq 15$, and $c(H) \leq s$. It follows in this case that $\Phi(H) \geq 0$; please refer to the figure Case 4.21–(D) in Figure 11 in Section 7.

Case 4.22 *All neighbors of the middle vertices in S are of degree 2.*

Let u'', v'', w'' be the degree-2 neighbors of u', v', w', respectively, and let u''', v''', w''' be the non-steeple neighbors of u'', v'', w'', respectively. We distinguish a few subcases.

Subcase 4.22.1. At least one vertex in $\{u''', v''', w'''\}$ is of degree 1.

Suppose that u''' is of degree 1; please refer to the figure Subcase 4.22.1 in Figure 12 in Section 7. Let $S_H = \{u'''\}$ and let H the subgraph of G_2 induced by S_H and its neighbor u''. Then there is one fringe edge to H, namely (u'', u'). We have $n(H) \leq 2, nt(H) \leq 2, e^+(H) \geq 2$, $c(H) \leq s/4$, and $G_2 - V(H)$ does not contain a bad component. Therefore, $\Phi(H) \geq 0$ in this case, and Case 4.18 above applies to the steeple S since $d(u') = 2$ in the resulting graph.

We can assume now that all vertices u''', v''', w''' are of degree at least 2.

Subcase 4.22.2. Two vertices in $\{u''', v''', w'''\}$ are identical.

Suppose that $u''' = v'''$. Let $S_H = \{u'', v'', w'', x, v\}$, and let H be the subgraph of G_2 induced by the vertices in H and their neighbors. There are at most three fringe edges to H: at most 2 incident on w''' and at most 1 incident on $u''' = v'''$. Since **Peel** does not apply to $G_2 - V(H)$, if $G_2 - V(H)$ contains bad components, then by Lemma 4.16, it must contains exactly one bad component B that is a leaf-block; moreover, B must be a C_5 block. Please refer to the figure Subcase 4.22.2-(A) in Figure 12 in Section 7. In this case $G_2[V(H) \cup V(B)]$ forms a separate component C of G_2. It is easy to see that there is an independent set for C consisting of S_H plus two vertices from B, and we can apply the operation to C with $n(C) \leq 17, nt(C) \leq 14, e^+(C) \geq 23$, and $c(C) = 0$. In this case we have $\Phi(C) \geq 0$. Suppose now that $G_2 - V(H)$ does not contain a bad component. Please refer to the figure Subcase 4.22.2-(B) in Figure 12 in Section 7. Since there are at most 3 fringe edges to H, $c(H) \leq 3s/2$. Note in this case that it is possible that a fringe edge to H is incident on a neighbor of a top vertex of a type-II steeple. We have $n(H) \leq 12, nt(H) \leq 9$, and $e^+(H) \geq 16$. It follows in this case that $\Phi(H) \geq 0$.

We can assume now that all vertices u''', v''', w''' are distinct and of degree at least 2. Let $S_H = \{u'', v'', w'', x, v\}$, and let H be the subgraph of G_2 induced by S_H and its neighbors. Since there are at most 6 fringe edges to H, and since **Peel** is not applicable to $G_2 - V(H)$, by Lemma 4.16, there can be at most two bad components in $G_2 - V(H)$. Moreover, if there are exactly two bad components, then each component is a single C_5 block.

Subcase 4.22.3. There are exactly two bad components in $G_2 - V(H)$.

Let B_1 and B_2 the single C_5 blocks in $G_2 - V(H)$, and note that there are exactly three fringe edges to H that are incident to vertices in B_1, and similarly for B_2. By Corollary 4.14, $\phi^-(B_1) = \phi^-(B_2) = -1/7$. By Lemma 4.12, we have $\phi(G_2[V(H) \cup V(B_1) \cup V(B_2)]) \geq \phi(H) + \phi^-(B_1) + \phi^-(B_2)$.

We have $n(H) \leq 13, nt(H) \leq 10, e^+(H) \geq 21$, and $c(H) = 0$. This gives $\phi(H) > 2/7$; therefore, $\Phi(G_2[V(H) \cup V(B_1) \cup V(B_2)]) \geq 0$. Please refer to the figure Subcase 4.22.3 in Figure 12.

Subcase 4.22.4. There is exactly one bad component B in $G_2 - V(H)$.

At least 3 fringe edges to H are incident on B. Suppose that there are t fringe edges to H that are not incident on B, where $0 \leq t \leq 3$. Note that these edges are incident on u''', v''', w'''. Then $c(H) \leq ts/2$. By Corollary 4.14, we have $\phi^-(B) = -1/7$. By Lemma 4.12, we have $\phi(G_2[V(H) \cup V(B)]) \geq \phi(H) + \phi^-(B)$. We have $n(H) \leq 13, nt(H) \leq 10, e^+(H) \geq 18 + t$, and $c(H) \leq ts/2$. It is easy to verify now that for any value of $0 \leq t \leq 3$, $\phi(H) > 1/7 + ts/2$, and hence $\Phi(G_2[V(H) \cup V(B)]) \geq 0$. Please refer to the figure Subcase 4.22.4 in Figure 12.

Subcase 4.22.5. There are no bad components in $G_2 - V(H)$.

Since each of u''', v''', w''' has degree at least 2, we have $e^+(H) \geq 15 + t$, where $2 \leq t \leq 6$. Moreover, H can have at most t fringe edges, and hence, $c(H) \leq ts/2$. We have $n(H) \leq 13, nt(H) \leq 10, e^+(H) \geq 15 + t$, and $c(H) \leq ts/2$. It is easy to verify that $\Phi(H) \geq 0$ for $2 \leq t \leq 6$. Please refer to the figure Subcase 4.22.5 in Figure 12.

Observe that none of the operations above affects the remaining steeples in the resulting graph.

Operations on type-II steeples

Let S be a type-II steeple, and let u, v, w be its triangle vertices, u', v', w' be its middle vertices, and x, y, z be its top vertices, as illustrated in Figure 7 in Section 7.

Case 4.23 *Each of x, y, z is of degree 3.*

Let $S_H = \{u', v', w'\}$ and let H be the subgraph of G_2 induced by S_H and its neighbors. Since no edge exists between x, y, z, there are exactly three fringe edges to H incident on x, y, z. Since **Peel** does not apply, by Lemma 4.16, there can be at most one bad component in $G_2 - V(H)$ that must be a single C_5 block. However, by part (v) of Proposition 4.10, this is impossible. We conclude that no bad components exist in $G_2 - V(H)$. Since the distance between any two type-II steeples is at least 3, no fringe edge to H can be incident on a neighbor of a top vertex in a type-II steeple in $G_2 - V(H)$. Therefore, $c(H) \leq 3s/4$. We have $n(H) \leq 9, nt(H) \leq 6$, and $e^+(H) \geq 15$. It follows that $\Phi(H) \geq 0$. Please refer to the figure Case 4.23 in Figure 13.

We can now assume that at least one vertex in x, y, z is of degree 2.

Case 4.24 *At least two vertices in x, y, z are of degree 2.*

Suppose that x and y have degree 2. Let $S_H = \{x, y, u\}$ and let H be the subgraph of G_2 induced by S_H and its neighbors. Then there are exactly 2 fringe edges to H incident on z. Clearly, there are no bad components in $G_2 - V(H)$ and $c(H) \leq s/2$ (none of the fringe edges can be incident on a neighbor of a type-II steeple in $G_2 - V(H)$ since the distance between any two type-II steeples is at least 3). We have $n(H) \leq 8, nt(H) \leq 5, e^+(H) \geq 12$, and $\Phi(H) \geq 0$. Please refer to the figure Case 4.24 in Figure 13.

We can now assume that exactly one vertex in x, y, z is of degree 2, say vertex x. Let y', z' be the neighbors of y, z, respectively, that are not vertices of S, and note that by part (iv) of Proposition 4.10, y' and z' are distinct. Let $S_H = \{x, y, z, u\}$, and let H be the subgraph of G_2

induced by S_H and its neighbors. Note that there are at most 4 fringe edges to H: at most 2 incident on y' and at most 2 incident on z'. Therefore, there can be at most one bad component B in $G_2 - V(H)$.

Case 4.25 *There exists a bad component B in $G_2 - V(H)$.*

In this case at least three fringe edges to H are incident on B. Since there are at most 4 fringe edge to H, we have $c(H) \leq s/2$. By Corollary 4.14, we have $\phi^-(B) = -1/7$. By Lemma 4.12, we have $\phi(G_2[V(H) \cup V(B)]) \geq \phi(H) + \phi^-(B)$. Now $n(H) \leq 11, nt(H) \leq 8, e^+(H) \geq 17$, and $c(H) \leq s/2$. It follows that $\phi(H) > 1/7 + s/2$ and $\Phi(G_2[V(H) \cup V(B)]) \geq 0$. Please refer to the figure Case 4.25 in Figure 13.

We can now assume that there are no bad components in $G_2 - V(H)$.

Case 4.26 *At least one of y', z' is of degree 1.*

Suppose that y' is of degree 1. Let $I = \{y', u', v', w'\}$ and let H_I be the subgraph of G_2 induced by the vertices in I and their neighbors. Then there is one fringe edge to H_I, namely the edge (z, z'). Therefore, $G_2 - V(H_I)$ does not contain any bad components. Now $n(H_I) \leq 10, nt(H_I) = 7, e^+(H_I) \geq 14$, and $c(H_I) \leq s/4$. It follows in this case that $\Phi(H_I) \geq 0$. Please refer to the figure Case 4.26 in Figure 13.

Now we can assume that both y', z' are of degree at least 2.

Case 4.27 *There is at least one fringe edge to H.*

Let t be the number of fringe edges to H. If $t = 1$, then since the degree of each of y', z' is at least 2, y' and z' must be adjacent, and $e^+(H) = 16$, $c(H) \leq s/2$ in this case. If $t \geq 2$, then $e^+(H) \geq 14 + t$ and $c(H) \leq ts/2$. Since $n(H) \leq 11, nt(H) \leq 8$, it follows in both cases that $\Phi(H) \geq 0$ (note that no bad components exist in $G_2 - V(H)$ at this point). Please refer to the figures Case 4.27-(A) and Case 4.27-(B) in Figure 13.

If none of the above cases applies, we must have:

Case 4.28 *y' and z' are of degree 2 and are adjacent.*

By part (iv) of Proposition 4.10, at the beginning of the third phase each neighbor of a top vertex of a type-II steeple is of degree 3. Therefore, at the beginning of the third phase both y' and z' were of degree 3. It follows that each of y' and z' lost an incident edge due to a previous operation in the third phase, and hence a cost of $s/2$ was associated with the removal of each of these edges. Consequently, we have a surplus of $s/2$ for each of y' and z', and hence a total surplus of s. Now $n(H) \leq 11, nt(H) \leq 8, e^+(H) \geq 15$, and we have $\Phi(H) + s \geq 0$. Please refer to the figure Case 4.28 in Figure 13.

4.4 Putting all together

Theorem 4.29 *Let G be an obstacle-free graph with $\Delta(G) \leq 3$. Then $\alpha(G) \geq n(G)/3 + nt(G)/42$.*

Proof. Let G_1 be the graph resulting after the first phase, and let G_2 be the graph resulting after applying the second phase to G_1. To show that $\alpha(G) \geq n(G)/3 + nt(G)/42$, by part (vii) of Proposition 4.10, it suffices to show that $\alpha(G_2) \geq n(G_2)/3 + nt(G_2)/42$.

If G_2 contains a bad component C, then since every triangle in G_2 is contained in a steeple, which is a 2-connected subgraph, C is triangle-free. By [11], we have $\alpha(C) \geq 5n(C)/14 = n(C)/3 + n(C)/42$. By additivity, it suffices to show that $\alpha(G_2 - V(C)) \geq n(G_2 - V(C))/3 + nt(G_2 - V(C))/42$. Therefore, we can assume that every component of G_2 contains some triangle, and that G_2 does not contain bad components. Let G_3 be the graph resulting from G_2 after the third phase. Since none of the operations in the third phase introduces a bad component, and since the operations remove all triangles from G_2, G_3 is a triangle-free graph that does not contain bad components. Therefore, by [10], $\alpha(G_3) \geq (4n(G_3) - e(G_3))/7 = (23n(G_3) - 6e(G_3) + nt(G_3))/42$. (The last inequality is true because $nt(G_3) = n(G_3)$.) Now each operation in the third phase that removes a subgraph H satisfies $\Phi(H) \geq 0$. The operations in the third phase remove the subgraph $G_2 - V(G_3)$ from G_2. By additivity of the function Φ, it follows that $\Phi(G_2 - V(G_3)) \geq 0$, and hence $\phi(G_2 - V(G_3)) \geq 0$. Since each independent set S_H of a removed subgraph H consists of internal vertices to H, we have $\alpha(G_2) \geq (23n(G_2 - V(G_3)) - 6e^+(G_2 - V(G_3)) + nt(G_2 - V(G_3)))/42 + \alpha(G_3)$. Since $n(G_2) = n(G_2 - V(G_3)) + n(G_3)$, $e(G_2) = e^+(G_2 - V(G_3)) + e(G_3)$, and $nt(G_2) = nt(G_2 - V(G_3)) + nt(G_3)$, we have $\alpha(G_2) \geq (23n(G_2) - 6e(G_2) + nt(G_2))/42$. Since $\Delta(G_2) \leq 3$, we have $e(G_2) \leq 3n(G_2)/2$, and hence $\alpha(G_2) \geq (14n(G_2) + nt(G_2))/42 = n(G_2)/3 + nt(G_2)/42$.

This completes the proof. □

5 The kernel

Let (G, k) be an instance of IS-3. The validity of the next four reduction rules follows from Fact 3.1, Fact 3.2, Lemma 3.3, and Lemma 3.4, respectively. The validity of Reduction Rule 5.5 is easy to see.

Reduction Rule 5.1 *Let (u, v, w) be a triangle in G such that $d(u) = 2$. Then include u in the solution, and set $G := G - \{u, v, w\}$ and $k := k - 1$.*

Reduction Rule 5.2 *Let (u, v, w) and (p, v, w) be two triangles in G that share an edge (v, w). Then set $G := G - v$ (i.e., vertex v can be removed from G).*

Reduction Rule 5.3 *Let $T_1, \ldots, T_\ell$ be a cycle of triangles in G, where $T_i = (u_i, v_i, w_i)$ for $i = 1, \ldots, \ell$, u_i is adjacent to v_{i+1} for $i = 1, \ldots, \ell - 1$, and u_ℓ is adjacent to v_1. Then include vertices $\{v_1, \ldots, v_\ell\}$ in the solution, and set $G := G - \bigcup_{i=1}^{\ell} V(T_i)$ and $k := k - \ell$.*

Reduction Rule 5.4 *Let H be a subgraph of G that is a medium obstacle. Then include the set of vertices $\{x, y, v_2\}$ of H in the solution, and set $G := G - V(H)$ and $k := k - 3$.*

Reduction Rule 5.5 *Let C be a component in G that is a large obstacle. Then include a maximum independent set of C in the solution, and set $G := G - V(C)$ and $k := k - 4$.*

Definition 5.1 Call a graph *reduced* if none of Reduction Rules 5.1–5.5 applies to the graph.

Let G be a reduced graph. A *tree of triangles* in G is a set of triangles such that the subgraph of G induced by the vertices of the triangles in this set is connected. A tree of triangles is *maximal* if it is maximal under set containment. We have the following lemma:

Algorithm **Kernelize**

INPUT: *An instance* (G, k) *of* IS-3
OUTPUT: *An instance* (G', k') *of* IS-3

1. **if** $k \leq n(G)/4$ **then accept** the instance (G, k);
2. **Repeat until** none of Reduction Rules 5.1–5.5 applies to (G, k):
   pick the first Reduction Rule in Reduction Rules 5.1, ..., 5.5 that applies to (G, k) and apply it;
3. let (G', k') be the resulting instance;
4. **if** $k' \leq 141n(G')/420$ **then accept** the instance (G, k);
   else return the instance (G', k').

Figure 4: The algorithm **Kernelize**.

Lemma 5.1 *Let G be a reduced graph, and let $\mathcal{T}$ be nonempty maximal tree of triangles in G. Then the number of edges with one endpoint a vertex in a triangle in $\mathcal{T}$ and the other endpoint a nontriangle vertex in G is at least $|\mathcal{T}| + 2$.*

PROOF. The statement of the theorem follows by a standard inductive proof on the number of triangles in $\mathcal{T}$. □

Lemma 5.2 *Let G be a reduced graph. Then the number of nontriangle vertices $nt(G)$ satisfies $nt(G) \geq n(G)/10$.*

PROOF. By Lemma 5.1, there are at least $(|\mathcal{T}| + 2)$ edges between any maximal tree $\mathcal{T}$ and nontriangle vertices in G. The statement now follows by summing over all maximal trees in G, and noting that the number of vertices in any maximal tree $\mathcal{T}$ is $3|\mathcal{T}|$ (because G is reduced, and hence no two triangles share vertices/edges), and that every nontriangle vertex has degree at most 3. □

Consider the algorithm given in Figure 4.

Theorem 5.3 *Given an instance (G, k) of* IS-3*, the algorithm* **Kernelize** *either accepts the instance (G, k) correctly, or returns an equivalent instance (G', k') of* IS-3 *such that $n(G') \leq 420k'/141$. The running time of the algorithm* **Kernelize** *is $O(k)$.*

PROOF. Since $\Delta(G) \leq 3$, G is 4-colorable and $\alpha(G) \geq n(G)/4$. Therefore, if $k \leq n(G)/4$ then the algorithm **Kernelize** can accept the instance (G, k) directly. It follows that step 1 of the algorithm is correct.

Let (G', k') be the instance of IS-3 resulting from (G, k) after step 2 of the algorithm **Kernelize**. The validity of Reduction Rules 5.1–5.5 follows from Facts 3.1–3.2, Lemmas 3.3–3.4, and the fact that a component that is a large obstacle has a maximum independent set of size 4. Therefore, (G', k') is an instance of IS-3 that is equivalent to the instance (G, k). Since none of Reduction Rule 5.2, Reduction Rule 5.4, Reduction Rule 5.5 applies to G', G' does not contain any obstacles (see Figure 1). It follows that G' is obstacle-free. By Theorem 4.29, $\alpha(G') \geq n(G')/3 + nt(G')/42$. Since G' is a reduced graph, by Lemma 5.2 we have $nt(G') \geq n(G')/10$. It follows from the previous two statements that $\alpha(G') \geq 141n(G')/420$, or equivalently, $n(G') \leq 420\alpha(G')/141$. Therefore, if $k' \leq 141n(G')/420$, then G' has an independent set of size k', and equivalently G has an independent set of size k; therefore the algorithm **Kernelize** can accept the instance (G, k). If this is not the case, then the algorithm returns the instance (G', k') in which $n(G') < 420k'/141$.

To argue that the running time of the algorithm is $O(k)$, note that after step 1 of the algorithm we have $k > n(G)/4$, or equivalently, $n(G) < 4k$. Now that the size of the graph is $O(k)$, it is not difficult to see that step 2 of the algorithm can be implemented to run in $O(k)$ time with the help of some suitable data structures. As a matter of fact, it is not difficult to see that Reduction Rules 5.1, 5.2, and 5.5 can be implemented to run in $O(k)$ time overall (throughout the whole execution of step 2). Moreover, with the help of an auxiliary graph whose vertices correspond to the triangles of G and whose edges correspond to adjacent triangles in G, which can be created and maintained in $O(k)$ time, Reduction Rules 5.3 and 5.4 can also be implemented to run in $O(k)$ time overall.

This completes the proof. □

Corollary 5.4 *The* IS-3 *problem has a kernel of size at most* $420k/141 < 3k$ *that is computable in* $O(k)$ *time.*

6 Kernel lower bounds

A *vertex cover* in a graph G is a set of vertices in $V(G)$ such that each edge in $E(G)$ is incident on at least one vertex in this set. The VERTEX COVER problem on graphs of maximum degree at most 3, abbreviated VC-3, is defined as follows:

> VC-3. Given an undirected graph G with $\Delta(G) \leq 3$, and a nonnegative integer k, determine if G has a vertex cover of size at most k.

The upper bound results on the kernel size for IS-3 in Theorem 5.3 give a lower bound on the kernel size for VC-3.

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two dual parameterized problems.[3] The following result was shown in [5]:

Lemma 6.1 ([5]) *If* $\mathcal{Q}_1$ *has a kernel of size* c_1k *and* $\mathcal{Q}_2$ *has a kernel of size* c_2k*, then unless P=NP,* c_1 *and* c_2 *must satisfy* $(c_1 - 1)(c_2 - 1) \geq 1$.

Moreover, it was shown in [5] that the INDEPENDENT SET and the VERTEX COVER problem are dual problems. It follows that the restrictions of INDEPENDENT SET and VERTEX COVER to graphs of maximum degree at most 3 are dual problems. Based on the previous statement, Lemma 6.1, and on Theorem 5.3, we derive the following result:

Theorem 6.2 *Unless P=NP, the* VC-3 *problem does not have a kernel of size at most* $420k/279$.

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7 Figures

Figure 5: The graph on the left consists of a cycle of length t with t triangles attached to it, as shown in the figure, where $t \geq 4$ is an even integer; we have $nt(G) = tr(G) = n(G)/4$, $\lambda(G) = 0$, $e(G) = 3n(G)/2$, and $\alpha(G) = 3n(G)/8$. The graph on the right consists of a cycle of length $3t$ ($t \geq 2$) with t triangles attached to it such that the vertices of each triangle are attached to three consecutive vertices on the cycle; we have $nt(G) = n(G)/2, tr(G) = n(G)/6, \lambda(G) = 0, e(G) = 3n(G)/2$, and $\alpha(G) = 5n(G)/12$.

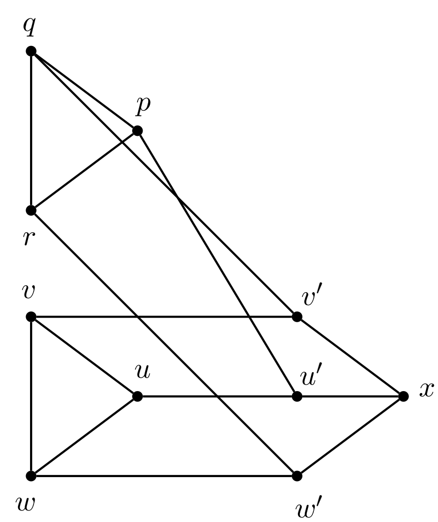


Figure 6: A *special component*.

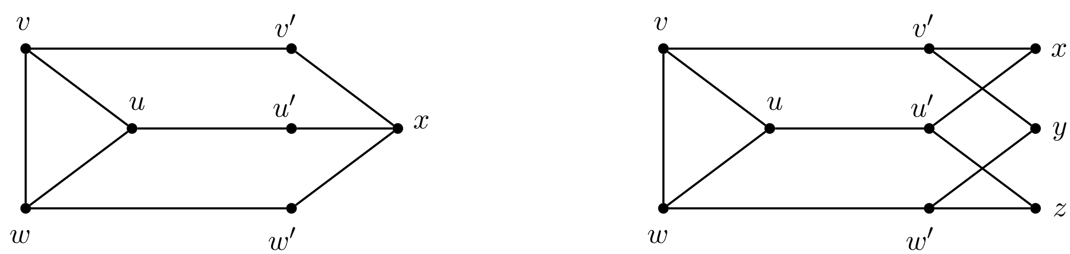
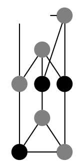
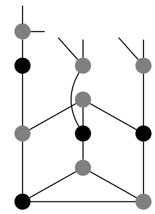


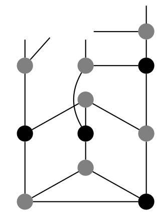
Figure 7: The *steeple graphs*. The graph on the left is referred to as a *type-I steeple* and that on the right as a *type-II steeple*. Note that no edges exist between any two vertices in $\{u', v', w'\}$ in both type-I and type-II steeples. Note also that the vertices u', v', w' in a type-I steeple could be either of degree 2 or 3; similarly, the vertices x, y, z in a type-2 steeple could be either of degree 2 or 3.



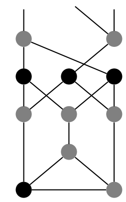
Op. 10: $n(H) = 8$, $nt(H) = 5, |S_H| = 3$.



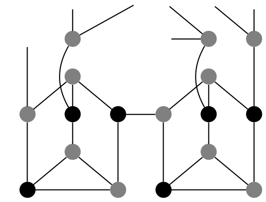
Op. 11: $n(H) = 11$, $nt(H) = 8, |S_H| = 4$.



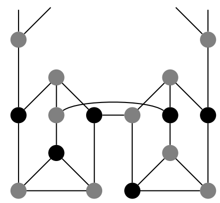
Op. 12: $n(H) = 11$, $nt(H) = 8, |S_H| = 4$.



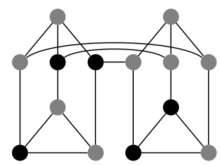
Op. 13: $n(H) = 11$, $nt(H) = 8, |S_H| = 4$.



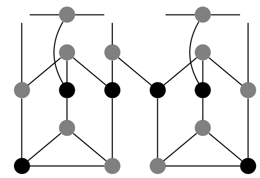
Op. 14: $n(H) = 17$, $nt(H) = 11, |S_H| = 6$.



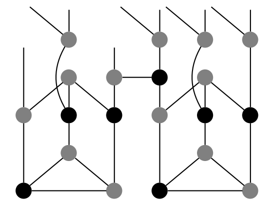
Op. 15: $n(H) = 16$, $nt(H) = 10, |S_H| = 6$.



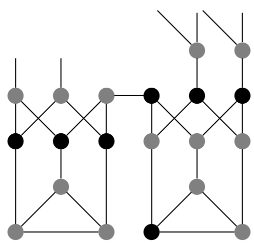
Op. 16: $n(H) = 14$, $nt(H) = 8, |S_H| = 5$.



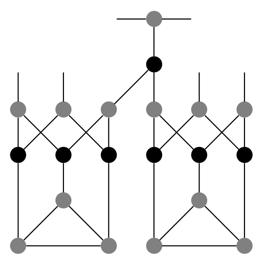
Op. 17: $n(H) = 17$, $nt(H) = 11, |S_H| = 6$.



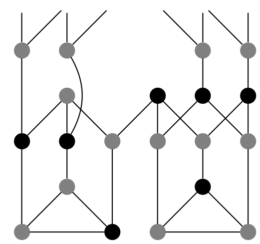
Op. 18: $n(H) = 20$, $nt(H) = 14, |S_H| = 7$.



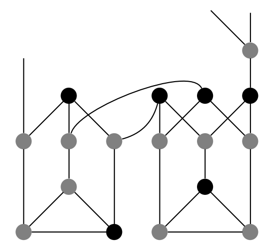
Op. 19: $n(H) = 20$, $nt(H) = 14, |S_H| = 7$.



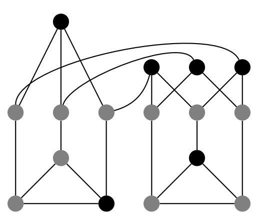
Op. 20: $n(H) = 20$, $nt(H) = 14, |S_H| = 7$.



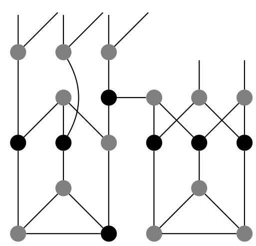
Op. 21: $n(H) = 20$, $nt(H) = 14, |S_H| = 7$.



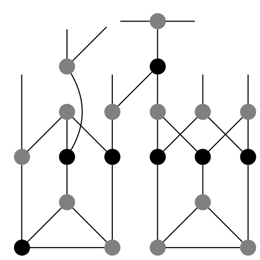
Op. 22: $n(H) = 17$, $nt(H) = 14, |S_H| = 6$.



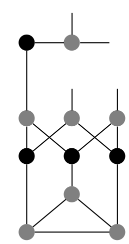
Op. 23: $n(H) = 16$, $nt(H) = 11, |S_H| = 6$.



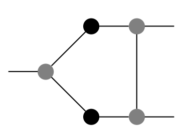
Op. 24: $n(H) = 20$, $nt(H) = 10, |S_H| = 7$.



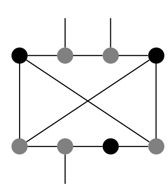
Op. 25: $n(H) = 20$, $nt(H) = 14, |S_H| = 7$.



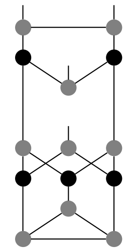
Op. 26: $n(H) = 11$, $nt(H) = 8, |S_H| = 4$.



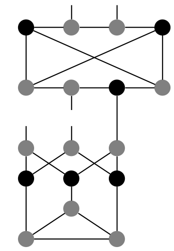
Op. 27: $n(H) = 5$, $nt(H) = 5, |S_H| = 2$.



Op. 28: $n(H) = 14$, $nt(H) = 11, |S_H| = 5$.



Op. 29: $n(H) = 8$, $nt(H) = 8, |S_H| = 3$.



Op. 30: $n(H) = 17$, $nt(H) = 14, |S_H| = 6$.

Figure 8: The second phase operations.

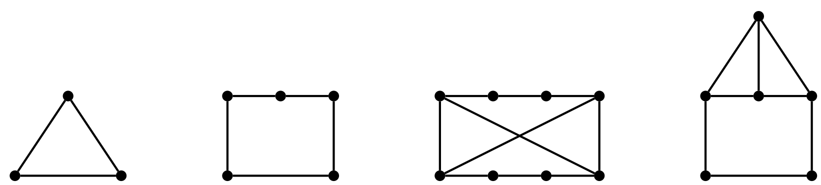
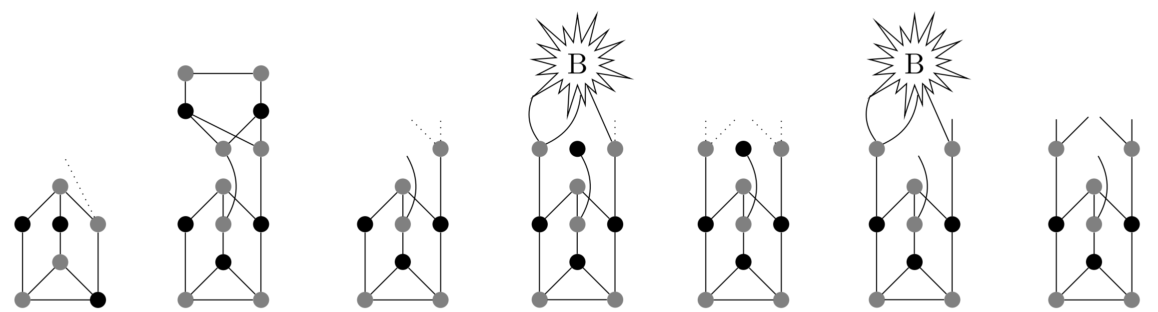
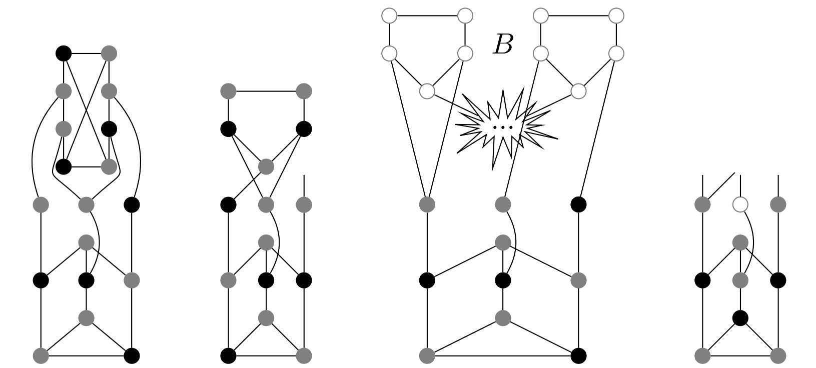


Figure 9: The difficult blocks.



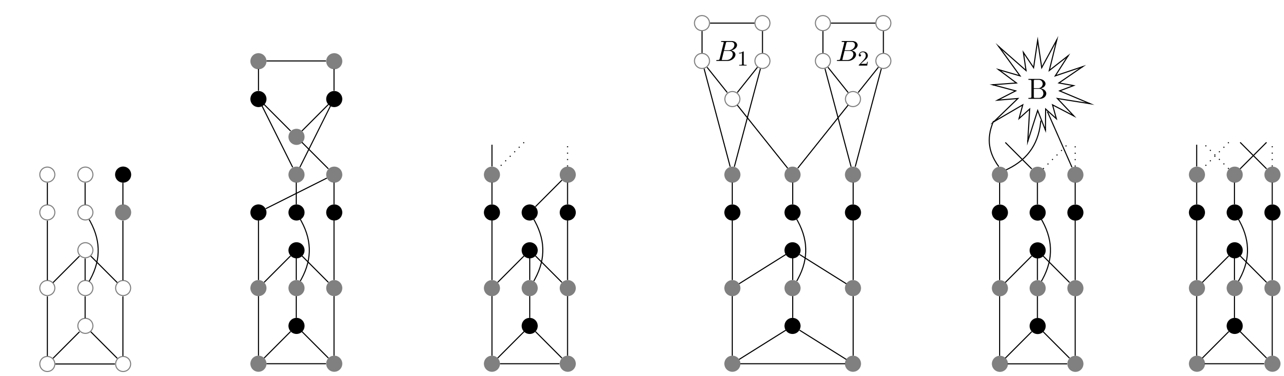
Case 4.17	Case 4.18-(A)	Case 4.18-(B)	Case **??**-(A)	Case **??**-(B)	Case 4.20-(A)	Case 4.20-(B)
$e^+(H) \geq 9$	$e^+(C) \geq 18$	$e^+(H) \geq 11$	$e^+(H) \geq 15$	$e^+(H) \geq 12$	$e^+(H) \geq 16$	$e^+(H) \geq 16$
$nt(H) \leq 4$	$nt(C) \leq 10$	$nt(H) \leq 5$	$nt(H) \leq 7$	$nt(H) \leq 7$	$nt(H) \leq 6$	$nt(H) \leq 6$
$n(H) \leq 7$	$n(C) \leq 13$	$n(H) \leq 8$	$n(H) \leq 10$	$n(H) \leq 10$	$n(H) \leq 9$	$n(H) \leq 9$
$c(H) \leq \frac{s}{4}$	$c(C) = 0$	$c(H) \leq \frac{3s}{4}$	$c(H) \leq \frac{s}{4}$	$c(H) \leq s$	$c(H) \leq \frac{s}{2}$	$c(H) \leq \frac{5s}{4}$
			$\phi^-(B) = -\frac{1}{7}$		$\phi^-(B) = -\frac{1}{7}$	

Figure 10: The third phase operations (Cases 4.17– 4.20). A dashed edge indicates that the edge may or may not be present.



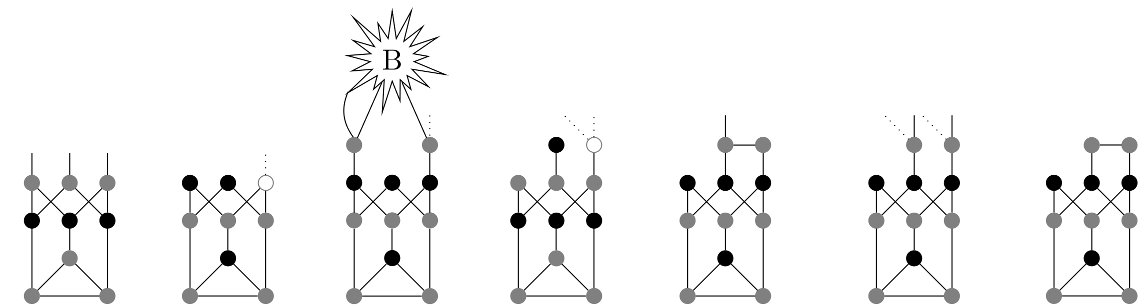
Case 4.21-(A)	Case 4.21-(B)	Case 4.21-(C)	Case 4.21-(D)
$e^+(C) \geq 26$	$e^+(H_I) \geq 21$	$e^+(H_I) \geq 16$	$e^+(H) \geq 15$
$nt(C) \leq 15$	$nt(H_I) \leq 12$	$nt(H_I) \leq 7$	$nt(H) \leq 6$
$n(C) \leq 18$	$n(H_I) \leq 15$	$n(H_I) \leq 10$	$n(H) \leq 9$
$c(C) = 0$	$c(H_I) \leq \frac{s}{4}$	$c(H_I) = 0$	$c(H) \leq s$
		$\phi^-(B) = -\frac{1}{7}$	

Figure 11: The third phase operations (Case 4.21).



Subcase 4.22.1	Subcase 4.22.2-(A)	Subcase 4.22.2-(B)	Subcase 4.22.3	Subcase 4.22.4	Subcase 4.22.5
$e^+(H) \geq 2$	$e^+(C) \geq 23$	$e^+(H) \geq 16$	$e^+(H) = 21$	$e^+(H) \geq 18 + t$	$e^+(H) \geq 15 + t$
$nt(H) \leq 2$	$nt(C) \leq 14$	$nt(H) \leq 9$	$nt(H) \leq 10$	$nt(H) \leq 10$	$nt(H) \leq 10$
$n(H) \leq 2$	$n(C) \leq 17$	$n(H) \leq 12$	$n(H) \leq 13$	$n(H) \leq 13$	$n(H) \leq 13$
$c(H) \leq \frac{s}{4}$	$c(C) = 0$	$c(H) \leq \frac{3s}{2}$	$c(H) = 0$	$c(H) \leq \frac{ts}{2}$	$c(H) \leq \frac{ts}{2}$
			$\phi^-(B_1) = \phi^-(B_2) = -\frac{1}{7}$	$\phi^-(B) = -\frac{1}{7}$	$(2 \leq t \leq 6)$
				$(0 \leq t \leq 3)$	

Figure 12: The third phase operations (Case 4.22). A dashed edge indicates that the edge may or may not be present.



Case 4.23	Case 4.24	Case 4.25	Case 4.26	Case 4.27-(A)	Case 4.27-(B)	Case **??**
$e^+(H) \geq 15$	$e^+(H) \geq 12$	$e^+(H) \geq 17$	$e^+(H) \geq 14$	$e^+(H_I) \geq 16$	$e^+(H) \geq 14 + t$	$e^+(H) \geq 15$
$nt(H) \leq 6$	$nt(H) \leq 5$	$nt(H) \leq 8$	$nt(H) \leq 7$	$nt(H_I) \leq 8$	$nt(H) \leq 8$	$nt(H) \leq 8$
$n(H) \leq 9$	$n(H) \leq 8$	$n(H) \leq 11$	$n(H) \leq 10$	$n(H_I) \leq 11$	$n(H) \leq 11$	$n(H) \leq 11$
$c(H) \leq \frac{3s}{4}$	$c(H) \leq \frac{s}{2}$	$c(H) \leq \frac{s}{2}$	$c(H) \leq \frac{s}{4}$	$c(H_I) \leq \frac{s}{2}$	$c(H) \leq \frac{ts}{2}$	$c(H) = 0$
		$\phi^-(B) = -\frac{1}{7}$			$(2 \leq t \leq 4)$	

Figure 13: The third phase operations (Cases 4.23– **??**). A dashed edge indicates that the edge may or may not be present.